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An estimate of the radial gradient of the toroidal magnetic field at the top of the Earth's core

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Abstract

In an $\alpha\omega$-type geodynamo, the toroidal magnetic field generated from the poloidal field through differential rotation can be sufficiently strong to make the Lorentz force comparable in strength with the Coriolis force. Thus the fluid flow at the top of the core should contain some information about the toroidal magnetic field. The magnetostrophic approximation is used in the momentum equation for fluid motion to relate the fluctuating part of the axisymmetric poloidal motion of the fluid with the radial gradient, $\partial B/\partial r$, of the steady part of the axisymmetric toroidal field at the core–mantle boundary (CMB). The former can be determined from a geomagnetic secular variation model using Braginsky's (Sov. Phys. JETP, 20: 1462–1471, 1965a) theory of the hydromagnetic dynamo. A geomagnetic secular acceleration model is then used to estimate $\partial B/\partial r$ at the CMB. The truncation level N for the geomagnetic field model is varied from three to six and consistent values of $\partial B/\partial r$ are only obtained for a range of colatitudes θ between 135° and 180°. It is seen that $|\partial B/\partial r|$ increases from zero at $\theta = 180°$ and attains a maximum near $\theta \approx 145°$ for $N = 3$ and $N = 4$, and $\partial B/\partial r$ is negative throughout this range of θ in all cases. The average value of $\partial B/\partial r$ over this range of θ is found to be around -4×10^{-6} T m^{-1}.

1. Introduction

Geomagnetic observations carried out over the last three centuries have provided time-dependent maps of the magnetic field at the core–mantle boundary (CMB) for this period (Bloxham and Jackson, 1992). These maps are based on the assumption that the mantle is an insulator. Barring the possibility of large conductivities in the middle and deeper parts of the mantle, where there are no reliable estimates of the conductivity at present, the source-free mantle approximation is considered to be adequate for the downward continuation of the magnetic field to the CMB (Benton and Whaler, 1983). Models of the secular variation of the magnetic field at the CMB have been used to estimate fluid flow near the top of the core assuming the core to be a perfect electrical conductor (Voorhies and Backus, 1985; Backus and Le Mouël, 1986; Lloyd and Gubbins, 1990; Benton and Celaya, 1991; Bloxham and Jackson, 1991). The core surface flow may have a small time-dependent part (Voorhies, 1993).

Dynamics of the fluid motion in the core is governed by pressure gradients, Coriolis, Lorentz and viscous forces and a buoyancy force which drives the convection (see, e.g. Gubbins and Roberts, 1987). In an $\alpha\omega$-dynamo model, the toroidal field generated from a poloidal field through differential rotation can be sufficiently strong to make the Lorentz force comparable with the Coriolis force. Hence the fluid flow at

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the top of the core can contain some information about the toroidal magnetic field. In this paper, Braginsky's (1965a) theory of the hydromagnetic dynamo is used together with the relevant momentum equation for fluid motion to show that the fluid motion can have a fluctuating axisymmetric component which can be related to the radial gradient of the steady part of the axisymmetric toroidal magnetic field near the CMB. Geomagnetic secular variation and secular acceleration models are then used to estimate this radial gradient for a certain range of colatitudes. Although Benton and Muth (1979) used a magnetostrophic vorticity balance in the momentum equation to estimate the radial gradient of the zonal magnetic field at some isolated points at the top of the Earth's core using geomagnetic field models, their assumptions required these points to be close to the geographic equator. In the present work, on the other hand, the assumptions made do not hold near the geographic equator, and estimates can be obtained only in the high-latitude regions.

2. Dynamo model

A steady self-consistent spherical hydromagnetic dynamo based on Braginsky's (1965b) nearly axisymmetric approximation has been studied by Fearn and Proctor (1987). These workers have split the equations governing the dynamo into axisymmetric and non-axisymmetric parts and computed the mean e.m.f. owing to the non-axisymmetric field and fluid flow by considering non- axisymmetric perturbations which can grow on a basic axisymmetric state. Then they proceeded in an iterative scheme to find a self-consistent steady axisymmetric magnetic field which can be maintained by the computed mean e.m.f. Temporal variations of the axisymmetric fields obviously cannot be treated through this approach. Information available about the Earth's magnetic field comprises not only the spatial structure of the poloidal part of the field outside the core but also its temporal variations. An attempt was made by Bhattacharyya (1992) to use the secular variation of the axisymmetric poloidal magnetic field on the CMB to estimate the azimuthal component E_ϕ of the mean e.m.f. generated by asymmetric fluid flow. However, so far there has been no effort to compare this with a theoretical computation of E_ϕ because of the complexity of the problem. In the present paper, information about the temporal variation of the poloidal field at the CMB is used to determine the vertical gradient of the steady part of the axisymmetric toroidal field at the top of the core on the basis of Braginsky's (1965a) theory of the hydromagnetic dynamo generalized to the non-stationary case together with the equations governing the fluid flow.

In this theory, the non-axisymmetric part $\boldsymbol{u}'$ of the fluid velocity in the core is represented by a superposition of waves propagating in the ϕ-direction and the axially symmetric velocity itself is assumed to consist of a slowly varying ($\partial/\partial t \approx \eta r_0^{-2}$, where η is magnetic diffusivity and r_0 is core radius) part and a rapidly oscillating part $\tilde{\boldsymbol{u}}$, which was termed 'oscillations' by Braginsky (1965a). Thus the total fluid velocity is written as

$$\mathbf{U}_{\text{tot}} = U\hat{\phi} + \mathbf{U}_P + \tilde{\mathbf{u}} + \mathbf{u}' \tag{1}$$

where $U\hat{\phi} + \boldsymbol{U}_P$ is the slowly varying axisymmetric part. It is assumed that $\tilde{u} \approx u' \approx U \mathrm{R}_\mathrm{m}^{-1/2}$ and $U_P \approx U \mathrm{R}_\mathrm{m}^{-1}$ with R_m as the magnetic Reynolds number defined by $\mathrm{R}_\mathrm{m} = U_M r_0/\eta$, where U_M is a characteristic value of U. The magnetic field is also similarly represented:

$$\mathbf{B}_{\text{tot}} = B\hat{\phi} + \mathbf{B}_P + \tilde{\mathbf{B}} + \mathbf{B}' \tag{2}$$

Both $\mathbf{u}'$ and $\mathbf{B}'$ have vanishing azimuthal averages as defined by Moffatt (1978). Likewise, time averages of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{B}}$ as defined by Braginsky (1965a) are also zero. Then, subtraction from the original induction equation

$$\frac{\partial}{\partial t}\mathbf{B}_{\text{tot}} = \nabla \times (\mathbf{U}_{\text{tot}} \times \mathbf{B}_{\text{tot}}) + \eta \nabla^2 \mathbf{B}_{\text{tot}} \tag{3}$$

of the doubly averaged (over ϕ and time) induction equation yields an equation for the short-term evolution of $\tilde{\mathbf{B}} + \mathbf{B}'$. The axially symmetric part of this resultant equation describes the short time scale variation of $\tilde{\mathbf{B}}$. Ignoring any slow changes in $\tilde{\mathbf{B}}$, the secular variation in $\tilde{\mathbf{B}}_P$ can then be described by the equation

$$\frac{\partial \tilde{\mathbf{B}}_P}{\partial t} = \left\{ \nabla \times \left[\tilde{\mathbf{E}} + \tilde{\mathbf{G}} + \tilde{\mathbf{u}} \times \tilde{\mathbf{B}}_P + \mathbf{U}_P \times \tilde{\mathbf{B}}_P - \eta \nabla \times \tilde{\mathbf{B}} \right] \right\}_P \tag{4}$$

where the mean e.m.f. values $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{G}}$ are given by

$$\tilde{\mathbf{E}} = \langle \mathbf{u}' \times \mathbf{B}' \rangle^{\phi} - \langle\langle \mathbf{u}' \times \mathbf{B}' \rangle\rangle \tag{5}$$

$$\tilde{\mathbf{G}} = \tilde{\mathbf{u}} \times \tilde{\mathbf{B}} - \langle \tilde{\mathbf{u}} \times \tilde{\mathbf{B}} \rangle^{t} \tag{6}$$

In Eqs. (5) and (6), $\langle\,\rangle^{\phi}$ denotes an azimuthal average, $\langle\,\rangle^{t}$ denotes averaging over 'fast' time, which is equivalent to averaging over the phases of the waves and oscillations, and $\langle\langle\,\rangle\rangle$ denotes a double average over both ϕ and time (Braginsky, 1965a). The e.m.f. $\tilde{\mathbf{E}}$ is responsible for giving rise to oscillations in the field owing to the presence of waves. In his theory, Braginsky neglected the term proportional to $\mathbf{U}_P$ and the diffusion term involving η in Eq. (4).

At the top of the core, $B = 0$ and $\tilde{B}_{\phi} = 0$ if the mantle is assumed to be an insulator. Also, if the CMB is assumed to be a free-slip, spherical boundary, $U_r = \tilde{u}_r = u'_r = 0$. Under these circumstances, Braginsky found that only oscillations of the field of order $\mathrm{R}_{\mathrm{m}}^{-3/2}$ can pass to the outside of the fluid core, and expressing $\tilde{\mathbf{B}}_P$ in terms of a vector potential $\tilde{A}\hat{\phi}$

$$\tilde{\mathbf{B}}_P = \nabla \times \left(\tilde{A}\hat{\phi}\right) \tag{7}$$

he determined that at the top of the core

$$\frac{\partial \tilde{A}}{\partial t} = \left[\tilde{\mathbf{u}}_P \times \mathbf{B}_P\right]_{\phi} \tag{8}$$

However, a consideration of the momentum equations governing $\tilde{\mathbf{u}}$ and $\mathbf{U}_P$ below shows that the contribution to $\partial\tilde{\mathbf{B}}_P/\partial t$ of the term proportional to $\mathbf{U}_P$ at the CMB cannot be neglected. Retaining this term, the short-term variation in $\tilde{\mathbf{B}}_P$ at the CMB is determined by

$$\frac{\partial \tilde{A}}{\partial t} = \left[\tilde{\mathbf{u}}_P \times \mathbf{B}_P + \mathbf{U}_P \times \tilde{\mathbf{B}}_P\right]_{\phi} \tag{9}$$

As discussed in the Introduction, the magnetic field strength can be large enough to produce a Lorentz force of strength comparable with that of the Coriolis force. From the earlier estimates of fluid flow at the CMB (Bloxham and Jackson, 1991) it seems reasonable to assume that flow speeds are generally very small compared with rotational speeds ($|U_{\mathrm{tot}}| \ll \Omega r \sin\theta$). Under these conditions, the magnetostrophic approximation should be valid, wherein viscous and inertial forces are neglected in the momentum equation for fluid flow. In addition, the usual Boussinesq approximation is made so that the fluid density in the outer core is taken to be a constant ρ_0 except in the buoyancy force. Then in a reference frame rotating with the mantle the momentum equation is given by

$$2\rho_0 \boldsymbol{\Omega} \times \mathbf{U}_{\mathrm{tot}} = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{B}_{\mathrm{tot}}) \times \mathbf{B}_{\mathrm{tot}} + \rho \mathbf{g} \tag{10}$$

where $\boldsymbol{\Omega} = \Omega\hat{z}$ and $\mathbf{g} = -g\hat{r}$. The local density ρ is given as a function of temperature by the equation of state:

$$\rho = \rho_0\left[1 - \alpha(T - T_0)\right] \tag{11}$$

where T_0 is some reference temperature and α is the thermal expansion coefficient. Variation in temperature itself is governed by the heat conduction equation (see, e.g. Gubbins and Roberts, 1987). In general, this whole system of equations along with the continuity equation for an incompressible fluid,

$$\nabla \cdot \mathbf{U}_{\mathrm{tot}} = 0 \tag{12}$$

and the induction Eq. (3) have to be solved subject to the boundary conditions that the mantle is a perfect electrical insulator and thermal conductor. Eq. (12) ensures that

$$\nabla \cdot \mathbf{U}_P = 0, \quad \nabla \cdot \tilde{\mathbf{u}}_P = 0 \quad \nabla \cdot \mathbf{u}' = 0 \tag{13}$$

just as

$$\nabla \cdot \mathbf{B}_P = 0, \quad \nabla \cdot \tilde{\mathbf{B}}_P = 0, \quad \nabla \cdot \mathbf{B}' = 0 \tag{14}$$

The fluctuating axisymmetric part of Eq. (10) is of the form

$$\begin{aligned} 2\rho_0 \boldsymbol{\Omega} \times \tilde{\mathbf{u}} &= -\nabla\tilde{p} + \frac{1}{\mu}\Big[(\nabla \times \overline{\mathbf{B}}) \times \tilde{\mathbf{B}} + (\nabla \times \tilde{\mathbf{B}}) \times \overline{\mathbf{B}} \\ &\quad + (\nabla \times \tilde{\mathbf{B}}) \times \tilde{\mathbf{B}} - \langle(\nabla \times \tilde{\mathbf{B}}) \times \tilde{\mathbf{B}}\rangle^{t} \\ &\quad + \langle(\nabla \times \mathbf{B}') \times \mathbf{B}'\rangle^{\phi} - \langle\langle(\nabla \times \mathbf{B}') \times \mathbf{B}'\rangle\rangle\Big] \\ &\quad - \rho_0 \alpha \tilde{T} \mathbf{g} \end{aligned} \tag{15}$$

where $\tilde{\mathbf{B}} = B\hat{\phi} + \mathbf{B}_P$, $\tilde{p}$ and $\tilde{T}$ are the fluctuating axisymmetric parts of the pressure and temperature, respectively, so that $\hat{\phi} \cdot \nabla \tilde{p} = 0$. To leading order in $R_m^{-1/2}$, the azimuthal component of Eq. (15) then yields

$$2\rho_0 \Omega \mu \tilde{u}_s \approx \left[\left(\nabla \times B\hat{\phi} \right) \times \tilde{\mathbf{B}}_P \right] \cdot \hat{\phi} \tag{16}$$

where (s, ϕ, z) denote cylindrical coordinates. On the CMB $(r = r_0)$ with $B = 0$ everywhere (16) reduces to

$$\tilde{u}_s = \frac{1}{2\rho_0 \Omega \mu} \frac{\partial B}{\partial r} \tilde{B}_r \tag{17}$$

Also at the CMB, $\tilde{u}_r = 0$, which implies that

$$\tilde{u}_z = -\tilde{u}_s \tan\theta \tag{18}$$

According to Braginsky's kinematic theory (1965a), in which $\tilde{u}$ and u' are assumed to be approximately $U R_m^{-1/2}$, at the CMB $\tilde{B}_r \approx R_m^{-3/2} B_M$, where B_M is a characteristic value of the axisymmetric, toroidal field B in the fluid core. Eq. (17), on the other hand, indicates that at the CMB, $\tilde{u}_P \not\approx R_m^{-1/2} U_M$ unless $\partial B/\partial r \gg B_M/r_0$. In terms of the Elsasser number $\Lambda = B_M^2 \sigma / \rho_0 \Omega$, which is usually considered as a measure of the ratio between magnetic and Coriolis forces, Eq. (17) yields

$$\tilde{u}_s \approx \frac{\Lambda}{2} \left(\frac{\partial B}{\partial r} \right) \frac{r_0}{B_M} R_m^{-5/2} U_M \tag{19}$$

For the Earth's core, if $\Lambda \approx 20$ (Gubbins and Roberts, 1987), $B_M \approx 50$ G and $B_M/r_0 \approx 0.01$ G km^{-1}. As the Lorentz force acting on the fluid involves spatial derivatives of the magnetic field, the Elsasser number alone may not be adequate to describe the strength of the Lorentz force vis-à-vis the Coriolis force. From Eq. (19) it is seen that the dimensionless quantity $(\partial B/\partial r)$ (r_0/B_M) also plays a role in determining the strength of the Lorentz force acting on the fluid. With $R_m \approx 200$, a value of $\partial B/\partial r \approx 40$ G km^{-1} yields $\tilde{u}_s \approx R_m^{-1/2} U_M$, which is consistent with Braginsky's assumption regarding $\tilde{u}$. However, this argument will have some consequences for the magnitude of U_P as well which need to be examined.

The azimuthal component of the steady axisymmetric part of Eq. (10) yields

$$2\rho_0 \Omega \mu U_s = \left\{ \left(\nabla \times B\hat{\phi} \right) \times \mathbf{B}_P + \langle (\nabla \times \tilde{\mathbf{B}}) \times \tilde{\mathbf{B}} \rangle^t + \langle\langle (\nabla \times \mathbf{B}') \times \mathbf{B}' \rangle\rangle \right\} \cdot \hat{\phi} \tag{20}$$

At the CMB, this simplifies to

$$2\rho_0 \Omega \mu U_s = \frac{\partial B}{\partial r} B_r + \langle \frac{\partial \tilde{B}_\phi}{\partial r} \tilde{B}_r \rangle^t + \langle\langle \frac{1}{r_0 \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta B'_\phi) B'_\theta + \frac{1}{r_0} \frac{\partial}{\partial r} (r B'_\phi) B'_r \rangle\rangle \tag{21}$$

With $\partial \tilde{B}_\phi / \partial r$ and $\partial B'_\phi / \partial r \approx R_m^{-1/2} \partial B / \partial r$, if on the CMB $\tilde{B}_r$ and $B'_r \approx R_m^{-3/2} B_M$, then the last two terms on the right-hand side of Eq. (21) may be neglected in comparison with the first term, yielding

$$U_s \approx \frac{1}{2\rho_0 \Omega \mu} \frac{\partial B}{\partial r} B_r \tag{22}$$

On the basis of the reasoning which was applied to Eq. (18) earlier, it is seen that U_s can be large enough to violate Braginsky's assumption regarding U_P. Hence there is need to reexamine the terms dependent on $\mathbf{U}_P$ which were neglected by Braginsky in arriving at Eq. (8). One of these is the term $(\mathbf{U} \times \tilde{\mathbf{B}}_P) \cdot \hat{\phi}$ which has been restored in Eq. (9). It is also necessary to consider the possible contribution to $\mathbf{B}'$ owing to $\mathbf{U}_P$, which was neglected by Braginsky and which can contribute to $\partial \tilde{B}_r / \partial t$ through $\tilde{E}_\phi$. At the CMB, with $u'_r = 0$, $\tilde{E}_\phi$ is given by

$$\tilde{E}_\phi = \langle\langle u'_\theta B'_r \rangle\rangle - \langle u'_\theta B'_r \rangle^\phi \tag{23}$$

The equation satisfied by $\mathbf{u}'$ is not considered here; it is simply assumed that $u' \approx R_m^{-1/2} U_M$. Retaining the terms involving $\mathbf{U}_P$ in the equation for waves $\mathbf{B}'_P$, one obtains the following up to first order in $R_m^{-1/2}$:

$$\frac{\partial \hat{B}'_r}{\partial t} + \frac{U}{s} B'_r - \frac{\partial U_r}{\partial r} \hat{B}'_r + \frac{U_\theta}{r} \frac{\partial \hat{B}'_r}{\partial \theta} = \frac{B}{s} u'_r; \tag{24}$$

where $\partial \hat{B}'_r / \partial \phi = B'_r$ as defined by Braginsky. At the CMB with $B = 0$ as also $u'_r = 0$, $B_r'^{(1)} = 0$ is a possible solution of Eq. (24), where $B_r'^{(1)} \approx R_m^{-1/2} B_M$. Then $\tilde{E}_\phi^{(2)}$, which is of order

$R_m^{-1/2} B_M U_M$, vanishes at the CMB. Likewise, it can be seen that at the CMB, $B_r'^{(2)} \approx R_m^{-1} B_M$ also satisfies an equation identical to Eq. (24) which has a solution $B_r'^{(2)} = 0$. As a consequence of this and $\tilde{u}_r = 0$ on the CMB, $\tilde{E}_\phi$ and $\tilde{G}_\phi$ do not contribute to $\partial \tilde{B}_r / \partial t$ to order $R_m^{-3/2}$.

The diffusion term on the right-hand side of Eq. (4) is neglected, as the oscillations are supposed to take place on a much shorter time scale than the diffusion time approximately r_0^2/η. In the present paper, some dynamical effects at the CMB have been added to the kinematic theory of Braginsky (1965a), but a proper study of the full scale dynamics has not been made in that the wave equations in the fluid which would give the space–time variation of $\mathbf{u}'$ also in a consistent manner have not been included. Such an approach has been outlined by Tough and Roberts (1968) in their treatment of the nearly symmetric hydromagnetic dynamo with a given non-axisymmetric body force. In their study, therefore, the time evolution of the body force itself has not been included, unlike the work of Fearn and Proctor (1987). In the present study, only some arguments have been put forward in the previous paragraph to demonstrate that the dynamical effects that have been added are not inconsistent, under the assumed conditions, with the derived form of the oscillations of the field $\partial \tilde{B}_r / \partial t$ obtained from Eq. (9). At the CMB, with $U_r = \tilde{u}_r = 0$, Eq. (9) yields the following for the short-term temporal evolution of $\tilde{B}_r$:

$$\frac{\partial \tilde{B}_r}{\partial t} = \frac{1}{r_0 \sin\theta} \frac{\partial}{\partial \theta} \left(\tilde{u}_z B_r + U_z \tilde{B}_r \right) \tag{25}$$

It can be seen from Eqs. (17), (18) and (22) that the term proportional to $U_z = -U_s \tan\theta$ in Eq. (25) makes a contribution which is identical to that of the term proportional to $\tilde{u}_z$. It is also clear that the variations in $\tilde{B}_r$ on a short time scale (much less than r_0^2/η) contain information about $\partial B / \partial r$.

3. Estimation of the radial gradient of B at $r = r_0$

A geomagnetic field model based on the usual spherical harmonic representation is used to extract $(\tilde{u}_z B_r + U_z \tilde{B}_r)$ at the CMB from Eq. (25). The radial component of the total main field, as described in Eq. (2), at the CMB is expressed as

$$B_r + \tilde{B}_r + B_r' = \sum_{n=1}^{N} \sum_{m=0}^{n} (n+1) \left(\frac{a}{r_0} \right)^{n+2} \left[g_n^m(t) \cos m\phi + h_n^m(t) \sin m\phi \right] P_n^m(\cos\theta) \tag{26}$$

where N is the truncation level, a is the radius of the Earth; g_n^m, h_n^m are Gauss coefficients and P_n^m are the Schmidt quasi-normalized associated Legendre functions. The axisymmetric part of Eq. (26) yields at the CMB

$$B_r + \tilde{B}_r = \sum_{n=1}^{N} (n+1) \left(\frac{a}{r_0} \right)^{n+2} g_n^0(t) P_n(\cos\theta) \tag{27}$$

As B_r is the steady part of the axisymmetric field, the time derivative of Eq. (27) gives for the secular variation in $\tilde{B}_r$ at the CMB

$$\frac{\partial \tilde{B}_r}{\partial t} = \sum_{n=1}^{N} (n+1) \left(\frac{a}{r_0} \right)^{n+2} \dot{g}_n^0(t) P_n(\cos\theta) \tag{28}$$

where $\dot{g}_n^0$ is the first derivative with respect to time of the Gauss coefficient g_n^0. To determine $(\tilde{u}_z B_r + U_z \tilde{B}_r)$ from Eq. (25), this is also expanded in terms of Legendre polynomials:

$$\tilde{u}_z B_r + U_z \tilde{B}_r = \sum_{n=0}^{\infty} E_n(t) P_n(\cos\theta) \tag{29}$$

On the CMB, at $\theta = 0$ and $\theta = \pi$, $\tilde{u}_z = \tilde{u}_r = 0$ and $U_z = U_r = 0$. Using either of these conditions, E_0 is determined to be

$$E_0 = \frac{2}{3} r_0 \left(\frac{a}{r_0} \right)^3 \dot{g}_1^0 \tag{30}$$

and for $n \neq 0$, E_n is given by

$$E_n = \frac{(n+2) r_0}{(2n+3)} \left(\frac{a}{r_0} \right)^{n+3} \dot{g}_{n+1}^0 - \frac{n r_0}{(2n-1)} \left(\frac{a}{r_0} \right)^{n+1} \dot{g}_{n-1}^0 \tag{31}$$

As a result of truncation of the series expansion for the poloidal field at $n = N$, all $E_n = 0$ for

$n > N+1$. According to Eqs. (17), (18) and (22), $(\tilde{u}_z B_r + U_z \tilde{B}_r)$ can be written as

$$\tilde{u}_z B_r + U_z \tilde{B}_r = -\frac{1}{\rho_0 \Omega \mu} \frac{\partial B}{\partial r} B_r \tilde{B}_r \tan\theta \tag{32}$$

Hence, once the left-hand side is known, $\partial B/\partial r$ can be estimated if B_r and $\tilde{B}_r$ can be separated in Eq. (27). To justify use of the expression given in Eq. (17) for $\tilde{u}_s$, based on a magnetostrophic balance, it has been assumed that at the CMB, $\partial B/\partial r \gg B_M/r_0$. However, the left-hand side of Eq. (32) is determined entirely from the observed short-period fluctuations of the main field. Therefore, validity of the above assumption can be tested only after $\partial B/\partial r$ has been estimated using Eq. (32).

In principle, an average over a time period much larger than the time scale of fluctuation of $\tilde{B}_r$ would yield B_r because $\langle \tilde{B}_r \rangle^t = 0$. In practice, the coefficients $g_n^0(t)$ are not available for a sufficiently long time period. To partly mitigate the problem, Eq. (32) is differentiated with respect to time and use is made of (29) to obtain

$$-\frac{1}{\rho_0 \Omega \mu} \frac{\partial B}{\partial r} \frac{\partial \tilde{B}_r}{\partial t} B_r \tan\theta = \sum_{n=0}^{N+1} \dot{E}_n(t) P_n(\cos\theta) \tag{33}$$

It can be seen from expressions (30) and (31) for the E_n values that $\dot{E}_n$ values can be computed from secular acceleration models of the geomagnetic field which give values of $\ddot{g}_n^0$ for a particular epoch. Hence, the radial gradient of the toroidal field at the CMB can be estimated from the relationship

$$\frac{\partial B}{\partial r} = -\rho_0 \Omega \mu \sum_{n=0}^{N+1} \dot{E}_n(t) P_n(\cos\theta) \Big/ \frac{\partial \tilde{B}_r}{\partial t} B_r \tan\theta \tag{34}$$

where $\partial \tilde{B}_r/\partial t$ is given by Eq. (28). The secular acceleration model of Langel et al. (1982) for epoch 1980, which gives values of $\ddot{g}_n^0$ up to $n = 6$, is used. The secular variation model coefficients for epoch 1980 are taken from Langel and Estes (1985). An estimate, albeit an approximate one, is also required for B_r. For this, averages of spherical harmonic coefficients for different epochs extending from 1550 to 1990, based on selected models listed by Barraclough (1978) and Langel (1992), are used to construct B_r. This problem was also addressed by Bloxham et al. (1989), who wanted to determine a stationary field pattern. They found that time-averaging over an interval of 265 years was not sufficient to remove all the time-dependent features. The estimate for B_r used in the present study also includes a part of $\tilde{B}_r$ and it is not possible to quantify the error owing to this in the estimate of $\partial B/\partial r$. Hence the results presented here must be considered as a preliminary estimate to be compared with estimates obtained through other methods (Benton and Muth, 1979; Stix and Roberts, 1984).

There are other constraints in estimating $\partial B/\partial r$ from Eq. (34). An examination of this equation shows that $\partial B/\partial r$ becomes infinite at those latitudes where either B_r or $\partial \tilde{B}_r/\partial t$ vanishes. Thus for a dipole field, it would not be possible to determine $\partial B/\partial r$ in the equatorial region using Eq. (34). These infinities of $\partial B/\partial r$ are unphysical and occur because at these latitudes the neglected contributions are more important. At $\theta = 0$ and $\theta = \pi$, even with non-vanishing B_r and $\partial \tilde{B}_r/\partial t$, both the numerator and denominator are zero. However, an application of L'Hospital's rule to the right-hand side of Eq. (34) shows that $\partial B/\partial r|_{r=r_0} = 0$ at $\theta = 0$ and $\theta = \pi$. According to Eqs. (17), (18) and (22), $\tilde{u}_z B_r = U_z \tilde{B}_r$. Hence Eq. (29) can be used to estimate $\tilde{u}_z$ itself:

$$\tilde{u}_z = \tfrac{1}{2} \sum_{n=0}^{N+1} E_n(t) P_n(\cos\theta)/B_r \tag{35}$$

Once again, $\tilde{u}_z$ can become infinite at those latitudes where B_r vanishes, which is in the neighbourhood of the equator. This unphysical behaviour is again a result of breaking down of assumptions, and $\partial B/\partial r$ cannot be estimated in such regions where $\tilde{u}_z$ becomes abnormally large. It will be recollected that in the momentum equation, inertial forces were neglected in comparison with Coriolis and other forces. An estimation of the acceleration $\partial \tilde{u}_z/\partial t$ using (35) shows that $|(\partial \tilde{u}_z/\partial t)/2\Omega \tilde{u}_s| \leqslant 10^{-1}$ for all colatitudes except for a narrow band of $\pm 5°$ around the equator. At $\theta = 0$ and π, because both the numerator and denominator vanish, L'Hospital's rule is applied to determine the ratio. If $\partial \tilde{u}_P/\partial t \approx \partial \tilde{u}_z/\partial t$, the smallness of this ratio justifies making the above assumption in the momentum equation except in the equatorial region. Truncation levels of $N=3$, 4, 5 and 6 have been used to estimate $\tilde{u}_z$, $\partial \tilde{u}_z/\partial t$ and $\partial B/\partial r$ at the CMB, keeping the various limitations listed above in mind.

4. Results and discussion

$\tilde{u}_z$ and $\partial \tilde{u}_z/\partial t$ computed from Eq. (35) are shown as functions of colatitude θ in Figs. 1 and 2, and the dependence of $\partial \tilde{B}_r/\partial t$ on θ as given by Eq. (28) is depicted in Fig. 3 for truncation levels $N=3$, 4, 5 and 6. For the secular variation and secular acceleration models used, $\tilde{u}_z$ and $\partial \tilde{u}_z/\partial t$ show more or less the same pattern of variation with θ for different truncation levels. The steep rise in $|\tilde{u}_z|$ and $|\partial \tilde{u}_z/\partial t|$ in the neighbourhood of $\theta = 90°$ is due to B_r approaching a value of zero in this region, and is unphysical, as explained in the preceding section. The plots of $\partial \tilde{B}_r/\partial t$ vs. θ, on the other hand, show a lot of variation for $N=6$ as compared with $N=3$, 4 and 5. The plots of $\partial \tilde{B}_r/\partial t$ for $N=4$ and 5 are identical because $\dot{g}_5^0 = 0$ for the secular variation

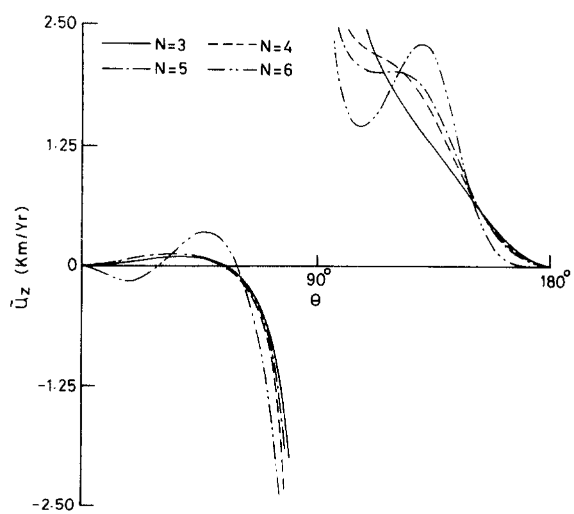

Fig. 1. $\tilde{u}_z$ computed from secular variation model for epoch 1980 as a function of colatitude for $N=3$, 4, 5 and 6.

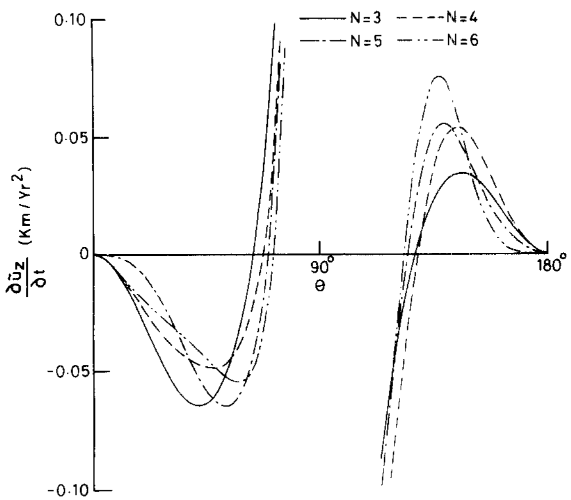

Fig. 2. $\partial \tilde{u}_z/\partial t$ computed from secular acceleration model for epoch 1980 as a function of colatitude for $N=3$, 4, 5 and 6.

model (Langel and Estes, 1985) used in the present calculation. For $N=3$, 4 and 5, $\partial \tilde{B}_r/\partial t$ changes sign in the neighbourhood of $\theta = 40°$ and

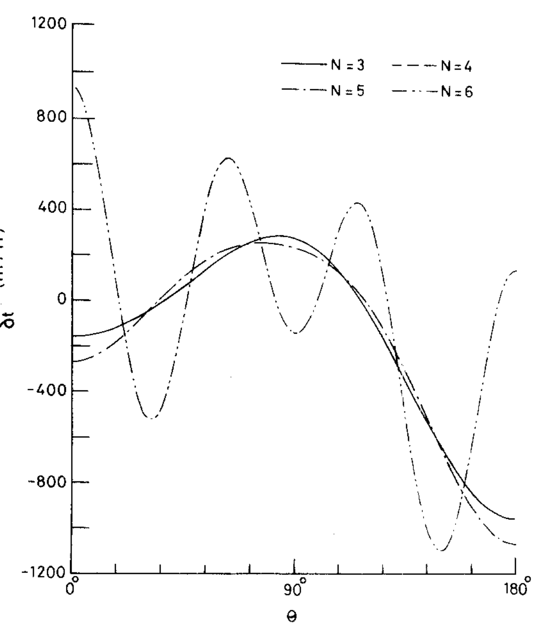

Fig. 3. Secular variation of the axisymmetric radial field at the core–mantle boundary, $\partial \tilde{B}_r/\partial t$, for epoch 1980 as a function of colatitude for $N=3$, 4, 5 and 6. The curves for $N=4$ and $N=5$ overlap because $\dot{g}_5^0 = 0$ for the secular variation model used.

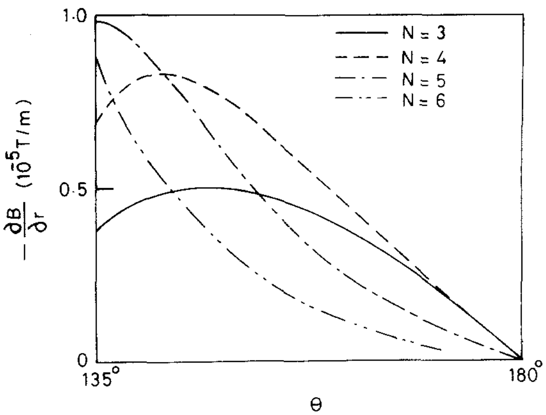

Fig. 4. Radial gradient of the steady, axisymmetric toroidal field B at the CMB, $\partial B/\partial r$, in the colatitude range $135° \leqslant \theta \leqslant 180°$.

$\theta = 120°$, and B_r itself changes sign near $\theta = 90°$. As a result of this kind of behaviour, there is a great deal of variation in the values of $\partial B/\partial r$ computed for the northern hemisphere using different values of N. As has been mentioned before, values of $\partial B/\partial r$ computed using expression (34) are unreliable for those latitudes where $B_r \approx 0$ or $\partial \tilde{B}_r/\partial t \approx 0$. Thus Eq. (34) provides more reliable estimate for $\partial B/\partial r$ in the region $135° \leqslant \theta \leqslant 180°$, than elsewhere, at least for $N = 3$, 4 and 5. In arriving at these estimates, the following values have been used for ρ_0, Ω and μ: $\rho_0 = 10^4$ kgm^{-3}, $\Omega = 7 \times 10^{-5}$ rads^{-1} and $\mu = 4\pi \times 10^{-7}$ Hm^{-1}. In Fig. 4, values of $\partial B/\partial r$ are plotted as a function of θ in this range, for $N = 3$, 4, 5 and 6. For $N = 6$, $\partial \tilde{B}_r/\partial t$ again changes sign in the vicinity of $\theta = 170°$, hence there is a break in the graph near this value of θ. Otherwise, over this range of latitudes in the southern hemisphere, $\partial B/\partial r$ is consistently negative and differs in magnitude by at most a factor of about two for the different truncation levels. Averages of the plotted values of $\partial B/\partial r$ are -35, -49, -41 and -29 G km^{-1} (10^{-7} T m^{-1}) for $N = 3$, 4, 5 and 6, respectively.

A method suggested by Benton and Muth (1979) for estimating the radial gradient of the zonal magnetic field at a few isolated points close to $\theta = 90°$ at the top of the Earth's core, using models of the main field and its secular variation, yielded $|\partial B/\partial r| \approx 10^{-6}$ T m^{-1} (10 G km^{-1}). It is expected that $\partial B/\partial r$ would depend on latitude and hence the present results are expected to differ from those of Benton and Muth (1979); however, it is worth noting that both results are of the same order of magnitude. On the other hand, the present results are almost two orders of magnitude larger than $\partial B/\partial r$ estimated at $r = r_0$, $\theta = 45°$ by Stix and Roberts (1984), and also have the opposite sign, as those workers predicted a positive sign for $\partial B/\partial r$ in the southern hemisphere. As far as the second point is concerned, it should be noted that Stix and Roberts (1984) obtained the same sign for $\partial B/\partial r$ throughout one hemisphere because they considered the following simple θ dependence of the toroidal field:

$$B\hat{\phi} = \nabla \times (T_C \hat{r}); \quad T_C = T_{C2}(r) P_2(\theta) \qquad (36)$$

The toroidal field may well have a more complicated dependence on θ than that entailed by Eq. (36), in which case there could be changes in the signature of $\partial B/\partial r$ within a hemisphere. As such, fluid flow on the CMB (approximately 10 km year^{-1}) obtained from inversion of geomagnetic secular variations at the CMB, when used directly in a magnetostrophic balance equation, would yield $|\partial B/\partial r| \approx 10^{-6}$ T m^{-1}. The problem in doing that is the non-uniqueness of the flow derived from such an inversion (Bloxham and Jackson, 1991). To resolve the non-uniqueness of the flow, the fluid motion at the CMB has been assumed to be either steady (Voorhies and Backus, 1985), or geostrophic (Backus and Le Mouël, 1986), or toroidal (Lloyd and Gubbins, 1990). Benton and Celaya (1991) have considered the time dependence of the geomagnetic field to be described by a low-degree even polynomial. One advantage of the present study is that the fluid motion is allowed to have a time-dependent component and the analysis itself is based on the magnetostrophic approximation. The inadequacies of the present approach have been discussed in the previous section. It is clear that the approximations made in arriving at the results given here fail in many instances, so that $\partial B/\partial r$ can be estimated for a severely limited range of θ. It is also not clear at present that the temporal variations of the geomagnetic field can be distinctly separated into short- and long-period fluctuations. As a result of all these uncertainities associated with the present results, it can only be stated that the estimates of $\partial B/\partial r$ presented here indicate that the toroidal field near the CMB can be very large compared with the poloidal field at the CMB. The magnetostrophic approximation used in the present work is also thus justified.

With the assumption of a strong ω effect taking place in the core and differential rotation dependent on θ as well as r, Matsushima and Honkura (1992) used geomagnetic field data to obtain a distribution for the axisymmetric toroidal field in which strong toroidal magnetic fields were present near the CMB. However, they have also stated that the method they have used for estimating differential rotation and the above distribution for the toroidal magnetic field may not be valid, in which case, using a different method, they concluded that the ω effect is not as effective as they had presumed and strong axisymmetric toroidal fields are not generated. The distribution of zonal toroidal magnetic field which Matsushima and Honkura (1992) obtained in their re-examination of fluid motion in the core is different from that obtained by Hollerbach and Jones (1993), who modelled the effect of the Earth's inner core on the geodynamo. These last researchers found the entire dynamo process and hence the axisymmetric azimuthal magnetic field also to be confined to the region outside the inner-core tangent cylinder. On the CMB, this cylinder has a radius about the poles of around 20°. In this context, it is perhaps worth pointing out that in Fig. 4, $|\partial B/\partial r|$ can be seen to gradually increase from a value of zero at the south pole to a maximum near 145° for $N=3$ and 4, which is definitely outside the inner-core tangent cylinder.

On the basis of preliminary estimates of $\partial B/\partial r$ presented here, it appears that a proper treatment of the problem based on the present approach may provide a better link between geomagnetic data and the geodynamo. This calls for an investigation into the growth of non-axisymmetric waves and resultant axisymmetric oscillations when the basic state is perturbed, as was done by Fearn and Proctor (1987), who, however, considered only a steady axisymmetric state. The problem can be linearized in the manner of those researchers to obtain the most unstable waves, and an iterative scheme can be followed to arrive at the final configuration of the steady part of the axisymmetric fields. In their study, Fearn and Proctor (1987) had to prescribe the mean toroidal field strength, the mean toroidal flow and the mean poloidal flow. By including some more of the dynamics as suggested here, it should be possible to obtain some more information from the equations themselves. Alternatively, the formalism developed by Tough and Roberts (1968) can be extended to include temporal variation of the body force itself through the evolution of the temperature T in Eq. (11) as given by the heat conduction equation. An attempt can then be made to solve the closed set of equations governing the geodynamo, by separating the axisymmetric and non-axisymmetric parts.

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