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SIGNAL PROCESSING IN GEOPHYSICS

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Many geophysical phenomena change with time and any collection of data measuring some aspect of such phenomena and ordered with respect to increasing time comprise a geophysical time series. If we have a reading for every moment of time we get a continuous series such as the photographic registration of variation in some component of earth's magnetic field. A plot of the time series shows immediately that mostly they are not merely strings of independent data; that most of the fluctuations are not violent and in general one can decipher constant fluctuation with set period such as the 11-year cycle in sunspot number. However, the measurement or recording of a time series can only cover a finite portion and any signal in the duration will generally appear noisy because of fluctuations either inherent in the signal or introduced by the measuring apparatus or the surroundings. Time series are not necessarily composed of a finite number of oscillations each with a discrete wavelength but of virtually infinite number of small oscillations spanning a continuous distribution of wavelengths. Spectrum is therefore a measure of the distribution of variance in a time series over a continuous domain of all possible wavelengths each arbitrarily close to the next ranging from an infinite wavelength (linear trend) to the Nyquist frequency ($1/2\Delta t$). It is now widely recognized that Spectral Techniques are rapidly becoming one of the most important statistical tools in the physical sciences. The most common reason for desiring a power spectral estimate is the supposition that the data might contain 'hidden' periodicities which would be more readily observable if the data were transformed to a spectral representation.

Many methods of hidden periodicities were suggested by workers such as Lagrange, Buys-Ballot and Whittaker but the best known was the Periodogram method made famous by Arthur Schuster. It consists of the function

$$I_n(w) = \frac{1}{n}\left[\left(\sum x_j \cos\frac{2\pi j}{\omega}\right)^2 + \left(\sum x_j \sin\frac{2\pi j}{\omega}\right)^2\right]$$

$I_n(w)$ will have a peak at $w=w_0$ if the data x_t $t=1,\ldots n$ contains a periodic term of period w_0 and there will be subsidiary peaks at $w = w_0 \pm 2w_0/n$ called side lobes. We will refer to those side lobes later in the discussion. Because of the enormous computation time needed to obtain

these Fourier transforms this approach was shelved till the advent in sixties of the Fast Fourier Transform routines.

In the next few sections we describe two of the widely used methods of spectral analysis and mention in brief about a radically different and new approach using the concept of Entropy in Information Theory.

The Blackman - Tukey Spectrum (Also called Correlation - Cosine Transform Procedure).

This method can be briefly described as follows:

Given a series of N equally spaced values 'Auto-correlation Coefficients' which are the quantities proportional to the ordinary linear correlation coefficients between a time series and the same time series after an interval of time, are computed for 'lags' of 0 to M units where $M < N$. Lag is the interval of time mentioned earlier. The cosine transform of these M + 1 lag correlation values are computed analogous to finding the Fourier transform of a continuous variable. The cosine transform yields M + 1 'Raw Spectral Estimates', the i-th value of which is a rough measure of the total variance in the original series that is contributed by wavelengths near the i-th harmonic of the fundamental wavelength of the analysis. The period P in the units of data interval is given by $2M/\Delta t \cdot i$. The raw estimates are then smoothed by a 3-term weighted moving average with weights equal to 0.25, 0.5, 0.25 in case of 'Hanning' and 0.23, 0.54 and 0.23 in the case of 'Hamming' method. The autocorrelation functions, also known as the mean lagged products, C_r are calculated as

$$C_r = \frac{1}{(n-r)} \sum x_i \, x_{i+r} - \frac{1}{(n-r)} \sum x_i \, \frac{1}{(n-r)} \sum x_{i+r}$$

The second term in the R.H.S. is for adjustment of the mean which makes the autocorrelation function proportional to the linear correlation coefficient. These autocorrelations may be considered the time-domain counterpart of the spectra in frequency domain. Valuable information regarding the time series can be obtained from the plot of autocorrelations with lag.

The 'Raw Spectral Estimates' are obtained as

$$H_0 = \frac{1}{2M} (C_0 + C_M) + \frac{1}{M} \sum_{r=1}^{M-1} C_r$$

$$H_M = \frac{1}{2M}\left(C_0 + (-1)^M C_M\right) + \frac{1}{M}\sum_{\lambda=1}^{M-1} C_\lambda$$

$$H_k = \frac{C_0}{M} + \frac{2}{M}\sum_{\lambda=1}^{M-1} C_\lambda \cos\frac{\pi k\lambda}{M} + \frac{1}{M}(-1)^k C_M$$

Smooth Spectral Estimates are then obtained from

$$S_0 = \frac{1}{2}(H_0 + H_1)$$

$$S_M = \frac{1}{2}(H_{M-1} + H_{M+1})$$

$$S_k = \frac{1}{4}(H_{k-1} + 2H_k + H_{k+1})$$

Instead of smoothing the raw spectral estimates as above, the autocorrelation functions can be multiplied by a 'Lag Window'. These two operations can be shown to be equivalent. For e.g. the lag window ½ (1 + cos πk/M) and the spectral window with weights 0.25, 0.50 and 0.25 are Fourier Transforms of each other.

It will be appropriate, at this stage, mentioning the necessity of 'Smoothing' the spectra by application of spectral windows or lag windows, to reduce leakage from side lobes. The effect of leakage is to smear the Power Spectral Density (PSD). It is easiest to visualise leakage in the form it takes in the continuous domain.

Let $C_x(\tau)$ be the autocorrelation of x(t). Then the PSD, $S_x(f)$ is obtained from

$$S_x(f) = \int_{-\infty}^{\infty} C_x(\tau)\, e^{-2\pi f\tau i}\, d\tau$$

Suppose now a PSD is computed with only a finite segment of $C_x(\tau)$

$$\hat{S}_x(f) = \int_{-T}^{T} C_x(\tau)\, e^{-2\pi f\tau i}\, d\tau$$

This is the type of calculation done in standard procedure.

$\hat{S}_x(f)$ can be written as

$$\hat{S}_x(f) = \int_{-\infty}^{\infty} U_T(\tau)\, C_x(\tau)\, e^{-2\pi f\tau i}\, d\tau$$

where

$U_T = 1 \quad -T < \tau < T$
$U_T = 0 \quad |\tau| > T$
} is the box car function

By the convolution theorem, if two functions are multiplied in the time domain, then their Fourier transforms are convolved in the frequency domain. Thus:

$$\hat{S}_x(f) = S_x(f) * U_T(f)$$

where $U_T(f) = 2\left(\sin 2\pi f T / 2\pi f\right)$, the Fourier transform of the box car function. For a sine series which is truncated, the power spectral density will be as shown in Fig.1

As T becomes larger the peak becomes narrower and higher. Because of the finite value of T, what would have been a delta function for infinite T has become a sin x / x function centred about f_0. Thus the power which was concentrated at a single point has spread out over a much broader range. It is this spreading of the power which is termed Leakage. The significant non-zero portions of the sin x/x function on either side of the main peak are referred to as Side Lobes. These will appear in one form or the other in all finite PSD calculations. The main purpose of the Windows (Lag or Spectral) is for the suppression of these side lobes. The windows generally broadens the main peak to a width of $4\Delta f$ while it suppresses the side lobes, substantially reducing leakage; the effect of application of a Hann window on raw spectral estimate is shown in Fig.2. Considerable efforts have been expended in design of these windows. Their properties vary depending on the weights used. Hann, Hamming, Bartlett, Parzen are some of the most common windows in B.T. spectral analysis. The choice of a particular window is best decided by the user depending on his requirements.

In the spectrum, various kinds of non-randomness will be revealed differently. For e.g., the spectrum of purely random variation tends to be rectangular in shape (all spectral estimates tend to have the same amplitude). This shape of the spectrum is often referred to as that of 'White Noise'. If a pure sine wave is contained in the time series, the spectrum will contain a relatively sharp peak at its appropriate wavelength. If a regular periodicity having non-sinusoidal shape is contained in the series, the spectrum will contain not only a peak at its basic wavelength but other peaks at wavelengths corresponding to one or more higher harmonics of its basic wavelength. If a quasi-periodicity or irregular rhythm is contained in the time series, the spectrum will indicate it by a relatively broad hump spanning an appropriate wide range of wavelengths. 27-day oscillation of the magnetic field presents itself as a good example. Finally, if the time series contains persistence, that is, if each value of the series is influenced by its immediately preceding value, the

spectrum will be distorted across all wavelengths. In particular, the amplitude of the spectrum will tend to decrease from the longer to the shorter wavelengths and the spectrum is then said to resemble that of 'Red Noise'.

From the manner of calculations of the autocorrelation function, it is evident that we cannot calculate arbitrarily large number of them as the number of terms to be multiplied and summed becomes progressively small. Hence an optimum maximum lag (M) for each set of N data points must be preferred. The spectral resolution is directly proportional to M and the stability of each spectral estimate is inversely proportional to M. The degrees of Freedom (DF) is approximately given by 2N/M. Thus a trade-off between resolution (large M) and stability of estimates (small M) must be achieved. The generally adopted criterion is M=N/4 or N/5. This aspect of the Blackman-Tukey spectrum is considered one of its drawbacks.

Test of Significance of Spectral Estimate:

As in other statistical techniques, we should be in a position to say with certain degree of confidence that a spectral peak is significant or in other words, the peak is not due to random variation in the time series but is due to a genuine periodicity present. According to Tukey (1950) each spectral estimate is distributed as chi-square/DF. From a Chi-Square table the value of chi-square/DF can be computed for any confidence level (usually 95% or 99%). A spectral peak is then considered significant at a particular confidence level if the spectral density at that frequency is in excess of (χ^2/DF) times the continuum level in the neighbourhood of the peak. Since the estimate of the continuum level here is not derived mathematically but is to be estimated for each supposed spectral peak, WMO panel of experts in Climatology gave a mathematical derivation of the continuum and the confidence levels for cases when the spectrum of is of 'White Noise' or the 'Red Noise' type, referred to earlier. (WHO Technical Note No. 79). The 'White Noise' spectrum is characterised by the fact that the lag-one serial correlation coefficient does not differ from zero by a statistically significant amount and the latter is characterised by the relation

$$r_2 \simeq r_1^2 \quad ; \quad r_3 \simeq r_1^3 \text{ etc.}$$

The continuum can be evaluated approximately from the relation

$$S_k = \bar{S}\left[\frac{(1-r_1^2)}{(1+r_1^2-2r_1\cos\frac{\pi k}{m})}\right]$$

where $\bar{S}$ is the average of all the M + 1 'raw' spectral estimates. An illustration of application of this technique is given in Fig.3. Of course, by far the easiest and surest test regarding the significance and reality of a spectral peak and its corresponding periodicity is to look for its presence in different samples of the data obtained from different epochs of the geophysical phenomenon.

Digital Filters in Spectral Analysis:

Since any spectrum spans the region Zero to Nyquist frequency it is evidently futile to expect much useful information from both ends of the spectrum simultaneously unless very large number of lags are used, which in turn means a large input of data. Alternatively, one can study the spectrum by choosing appropriate bands of interest and eliminating or reducing the effects of significant periodicites present outside the band. The appropriate remedy is to filter out the undesired frequencies at an early stage. The digital filters which serve such purposes can be broadly classified into (1) LOW PASS (2) BAND PASS and (3) HIGH PASS Filters. Each functions what the name implies. The first passes all low frequencies upto the desired frequency, second allows a preset band of frequencies only and the third allows all from a stipulated frequency to the Nyquist frequency. The filtering is usually achieved by means of 'sliding' a series of weights which determine the actual transfer function of the filter along the data. Behannon and Ness (1966) discuss in detail the design and application of numerical filters for geophysical data analysis.

Spectral Analysis via Fast Fourier Transform.

In the Blackman Tukey approach, the infinite set of which the sample formed a part, is assumed to be zero outside the sample i.e. the autocorrelations were truncated and assumed zero beyond the lag M. The magnitude of error caused by this truncation is somewhat reduced by smoothing the spectral estimate but the error increases as the data length becomes shorter and the relative power of any apparent periodicities become larger due to the problem of leakage. It is usually necessary that the length of data be many times longer than the greatest period of interest. The principal reason for popularity of this approach has been its computational simplicity. This has now largely been superceded by the DFT scheme which is generally considered more accurate but this was

not computationally feasible till the advent of the fast Fourier transform routines. Here the infinite set is considered periodic with one period equal to the available length of data. When this scheme is combined with proper tapering or smoothing it usually yields spectral estimates which are very accurate except in the low frequency portion of the spectrum when periods become comparable to the original length of data. Thus the segment of data should be long enough so that series errors occur well below the frequency range of interest. Still the required data length is less than that needed for comparable results using B.T. approach. Since the data length is normally an order of magnitude greater than the maximum lag in B.T., FFT can compute spectra which are more reliable at low frequencies.

If $X(j)$ $j=0, 1, \ldots\ldots N-1$ is a sequence of N finite value complex numbers, its Discrete Fourier Transform (DFT) is defined as

$$A(n) = \frac{1}{N} \sum_{j=0}^{N-1} X(j)\, e^{-2\pi i nj/N} \qquad i^2 = -1$$

$$W_N = e^{2\pi i/N}$$

$$= \frac{1}{N} \sum X(j)\, W_N^{-nj}$$

Fast Fourier Transform (FFT) is a method for efficiently calculating the DFT of a time series. It takes advantage of the fact that the calculation of the coefficients of DFT can be carried out iteratively which results in considerable saving of computer time. Specifically if the time series consists of $N = 2^n$ samples, then about $2nN = 2N \log_2 N$ arithmetic operations will be required to evaluate all the N associated DFT coefficients. In comparison with the number of operations required for calculating DFT coefficients with direct method (N^2) this number is very small. Fig.4 illustrates the drastic reduction in operation as the number of samples increases. About 5 seconds only are needed to get 8192 (2^{13}) DFT coefficients through an average computer as compared to about half-an-hour for the naive method.

FFT not only reduces computational time but also reduces the round-off errors associated with such computations. Vivid descriptions of how this speed is achieved in the computer memory using the symmetries of the sine and cosine functions are given in a series of papers which appeared in IEEE Trans. AU-15 and AU-17 in 1967 and1969.

If A(n) is the DFT of X(j) given as

$$A(n) = \frac{1}{N} \sum_{j=0}^{N-1} X(j) W_N^{-nj}$$

then, the raw power spectrum is computed as

$$P(n) = A(n)^2_{real} + A(n)^2_{imag.}$$

The direct calculation is an alternative to the common auto-covariance approach but the two methods can be shown to be equivalent. The direct calculation using FFT is, however, much faster for long data sequences.

In practical application of FFT routines for power spectrum, we adopt certain smoothing of the original data series and again smooth the raw spectra.

Data Window:

This is a function of discrete time series by which the time series is multiplied before Fourier transformation. Since we use only a finite segment of the time series, the discontinuities at the end of the measured segment introduces a rapid signal change that is not consistent with the band-width of the measurement. The difficulties at the data string become apparent as oscillations or Gibbs phenomena if attempts are made to interpolate or filter the signals. In order to lessen the effects of these discontinuities we make the end regions have a smooth transition to the mean of the measured value by multiplying the sequence of data with weights of Data window. The choice of the window function is somewhat arbitrary but generally the tapering achieved by use of a cosine bell over the first and last 10% of data string is adequate.

The flow diagrams given in Figs. 5, 6 and 7 describe briefly the three common methods of spectral analysis using FFT routines.

It may be seen that in the last method - Segmental Averaging - the stability of the spectral estimates is achieved not by application of spectral window but by averaging. The advantage of this method is that it can be used with blocks of data that are not necessarily consecutive, it affords an opportunity to test the stationarity of the time series as we consider many samples from different epochs and one can combine the estimates in such a way that the 'best' estimate is not biased by noise and confidence limits for the 'best' estimates may be obtained. In an illustrative paper, Black (1970) has discussed in detail the two approaches of power spectrum analysis, the B.T. and FFT, and has given a method of obtaining the 'Signal' amplitude from the 'Total' amplitude which includes 'Noise' contribution. (see fig. 8)

A particular periodicity in the time series generally will give a significant signal at several adjacent frequencies. The amplitude of the periodic oscillation is then taken as $(\Sigma S_k^2)^{1/2}$ where the sum is taken over the estimates

comprising the peak. The standard deviation of each estimate is known and the probable error of the amplitude is then taken as

$$\delta S = \left[\frac{\sum S_k^2 \sigma_{S_k}^2}{\sum S_k^2} \right]^{1/2}$$

Cross Spectra, Coherence and Phase Difference

The discussion so far has been related only to a single time series. Very often it is illuminating and interesting to study simultaneously two time series of geophysical phenomena which may be related. The spectra derived from single time series may be termed Auto Spectra, analogous to Autocorrelations computed from the same time series. We can also derive spectra and discuss two time series say X_t and Y_t. It is by use of cross-spectra that we can measure the extent to which a frequency component of X_t is correlated to the same frequency of Y_t and how much the two components are out of phase.

Let $C(\ell) = \langle Y_i X_{i-\ell} \rangle_i$ $\quad \langle \rangle_i$ implying average over all i

$D(\ell) = \langle X_i Y_{i-\ell} \rangle_i$

denote the positive and negative parts of the cross-correlations between the two time series. The Transforms

$$Z_k = \delta_k \left\langle \left(\frac{D(\ell)+C(\ell)}{2} \right) \left(1 + \cos \frac{\pi \ell}{M} \right) \cos \frac{\pi k \ell}{M} \right\rangle_i \qquad \delta_0 = \delta_M = 1$$

$$W_k = \delta_k \left\langle \left(\frac{D(\ell)-C(\ell)}{2} \right) \left(1 + \cos \frac{\pi \ell}{M} \right) \sin \frac{\pi k \ell}{M} \right\rangle_i \qquad \delta_k = 1/2 \text{ otherwise}$$

where k is the frequency and ℓ is the lag are called the Co-spectrum and Quadrature spectrum respectively. The cross spectrum then, is defined as the complex quantity

$$CS_k = Z_k + i\, W_k$$

A measure of the correlation between the frequency components of the two processes is given by

$$R_k = \left[\frac{Z(k)^2 + W(k)^2}{X(k) Y(k)} \right]^{1/2}$$

This R(k) is called the Coherence at frequency k, which is similar to the square of the correlation coefficient between the two samples and is interpreted in a similar

way i.e. larger the value more close is the relation between the two components.

The angle Θ_k defined by

$$\tan \Theta_k = \frac{W(k)}{Z(k)}$$

gives the phase lead of the Y-record relative to the X-record.

If the coherence is small at frequency k then both Z(k) and W(k) will be small and the estimate of W(k)/Z(k) is likely to have very large variance. Thus points in the phase diagram corresponding to frequencies with low coherence will generally contain less useful information than points corresponding to points with high coherence. It may be noted, however, that fluctuations in coherence donot affect the interpretation of the phase diagram but merely alters the variance of the estimate of the phase.

Cross spectra coherence and phase difference for two time series can also be obtained using the FFT routines. The following describes the steps involved:

Let $(a_j + ib_j)$ and $(c_j + id_j)$ be the Fourier transforms of the two time series which have been modified by application of a data window. Then the raw cross-spectrum $G_{xy,j}$ is obtained as

$$G_{xy,j} = R_{xy,j} + i\, S_{xy,j}$$

$$= \left[(a_j c_j + b_j d_j) + i\,(b_j c_j - a_j d_j) \right]$$

where R is the Co-spectrum and S is the quadrature spectrum. The coherence and phase differences are defined similar to that described earlier. It should be noted that R and S must be smoothed along with auto-spectra before the coherence and phase differences are computed. It is also important that the two series must be aligned and must of the same length for meaningful results.

Maximum Entropy Spectral Analysis

In discussing about the spectrum derived using B.T. approach or FFT routines, we mentioned about the nature of extension of the data outside the sample considered and said that both the assumptions were not truly realistic. While we cannot say that either of the schemes yields 'erroneous' results, or 'incorrect' spectral estimates, we can recognize the objections concerning the manner in which the data has been extended outside the domain. What is required is an entirely different approach in which it is not necessary to presume the extension of data outside the sample.

The most interesting characteristic of the spectral density function is the amount of information it contains. The concept of information has been shown to be equivalent to the concept of statistical entropy. Entropy is useful in statistical mechanics where it is a measure of the randomness of a system or the uncertainty of our knowledge about the state of the system. In Information Theory, it measures the average information content in a message.

Since the extension of data outside the sample is not defined, the ambiguity associated with the unknown points should have no effect on the information displayed by the spectrum i.e. no change in entropy should accompany variations in the unknown values. Any variations in the unknown would cause a variation in the auto-correlation coefficients. Hence a suitable scheme requires that the entropy does not change with respect to variations in a.c. coefficients which is equivalent to maximizing the entropy w.r.t. these autocorrelation coefficients. This is the philosophy which was given shape to by Burg (1968) and later by Ulrych and others. Apart from the inconsistencies regarding extension of data outside the sample, the other problems that beset B.T. and FFT techniques are their inadequacy when record length equals the longest period of interest, necessity for window functions to avoid leakage and the independence of these window functions from the spectral estimates and commonly observed frequency shifts when the recording length is less.

$$\text{Entropy } H = \int_{-f_0}^{f_0} \log P(f)\,df \quad \text{where } f_0 = \frac{1}{2\Delta t},\ \text{Nyquist frequency}$$

$$= \int_{-f_0}^{f_0} \log \sum \phi(k) e^{-2\pi i f k \Delta t}\, df$$

If H is made stationary w.r.t. the unknown autocorrelations, then

$$\frac{\partial H}{\partial \phi_k} = 0 \quad \text{for } |k| \geq N+1$$

$$\int_{-f_0}^{f_0} \frac{e^{-2\pi i f k \Delta t}}{P(f)} = 0$$

i.e. the spectral density P(f) must be such that $P(f)^{-1}$ and $e^{-2\pi i f k \Delta t}$ must be orthogonal in the interval $-f_0$ to f_0. The condition is in terms of the inverse of the spectral density function. Since the inverse function is directly associated with the concept of mean square error prediction it is natural to expect a useful computing algorithm to be couched in terms of prediction error filter. We can also say that Maximum Entropy spectrum is formulated by determining P(f) that maximizes H subject to the constraint that the autocorrelation $C(\tau)$ is given by

$$C(\gamma) = \int_{-f_0}^{f_0} P(f)\, e^{2\pi i f \tau}\, df$$

The constraint is that the autocorrelation must be given by the Fourier transform of the power spectrum.

The spectral density which gives an entropy stationary with reference to the unknown autocorrelations and which is also consistent with the known autocorrelations can be shown to be given by

$$S(f) = \frac{P_{N+1}}{2f_N \left| 1 + \sum_{j=1}^{N} \gamma_j e^{-i2\pi f j \Delta t} \right|^2}$$

f_N is the Nyquist freq

where $\gamma_j = 1, \ldots\ldots n$ and $\gamma_0 = 1$ are the (N+1) prediction error filter coefficients and P_{N+1} is the mean output power when the filter is applied in both the forward and backward direction over the data. This is necessary to ensure statistical accuracy, even though theoretically such a procedure is simply redundant because the mean output power is independent of the direction in which the filter is applied. The finiteness of the data could cause the filter coefficients to exceed unity, whereas all should be less than or equal to one only. The problem of MESA then reduces to finding the Prediction Error Filter and the mean output power for the Filter. This is achieved by means of a recursive solution of the equation

$$\begin{bmatrix} \phi_0 & \phi_1 & \cdots & \phi_N \\ \phi_1 & \phi_0 & \cdots & \phi_{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N & \phi_{N-1} & \cdots & \phi_0 \end{bmatrix} \begin{bmatrix} 1 \\ \gamma_{N1} \\ \gamma_{N2} \\ \vdots \\ \gamma_{NN} \end{bmatrix} = \begin{bmatrix} \beta_N^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Essentially the method consists of two steps. First, the coefficients of successively longer filters are estimated by applying the equation in a recursive fashion to the data. Then, after the desired number of coefficients have been estimated, they are substituted in the equation

$$S(f) = \frac{\beta_N^2 \,(= P_{N+1})}{2f_N \left| 1 + \sum \gamma_{Nj} e^{-2\pi i f j \Delta t} \right|^2}$$

to obtain the corresponding power spectral estimate.

The recurrent relations can be shown to be

$$\gamma_{N0} = 1$$

$$\gamma_{NK} = \gamma_{(N-1)K} + \gamma_{NN}\,\gamma_{(N-1)(N-K)} \qquad K = 1, 2, \ldots N-1$$

where γ_{NN} is computed from minimizing β_N^2

$$\beta_N^2 = \frac{1}{2(T-N)} \sum_{t=1}^{T-N} \left\{ \left[\gamma_{NK}\, x_{t+N-K}\right]^2 + \left[\gamma_{NK}\, x_{t+K}\right]^2 \right\}$$

$$\beta_N^2 = \beta_{N-1}^2 \left[1 - \gamma_{NN}^2\right]$$

Advantages and disadvantages of Maximum Entropy Spectral Analysis (MESA):

1) The procedure does not make any unrealistic assumption about data outside the sample domain.

2) The method can be used with success when data length is comparable to longest period of interest in contrast to the other two approaches.

3) The method eliminates bandwidth constraints because the frequency response of a digital filter can be computed for arbitrary values of frequency which is specially advantageous when analysing short records.

4) The method provides an unbiased estimate of spectrum shape because no fixed smoothing windows are applied.

5) In contrast to an unsmoothed FFT spectrum which gives good resolution but is very ragged, this method yields spectra which are smoother and more stable.

6) MESA is much more effective in detecting weak signals due to its high resolution capability.

Although the potential superiority of MESA over other spectral estimators in particular for short data lengths is well recognized, the usefulness of this approach is marred by the lack of criterion for choosing the length of the prediction error filter. Too short a length results in a highly smoothed estimate, obviating the resolution advantages of MEM whereas an excessive length introduces spurious detail into the spectrum.

In contrast to FFT, MESA routine will be computationally expensive.

The spectral peaks in MESA are not directly proportional to the square of the amplitude as is the case with the other

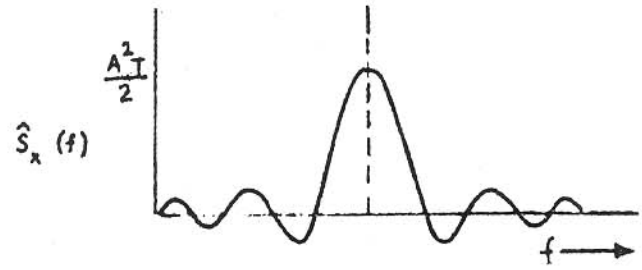


Fig. 1 Power spectral density of a sinusoid having a sin x/x form due to truncation. (after Otnes and Enochson, 1972)

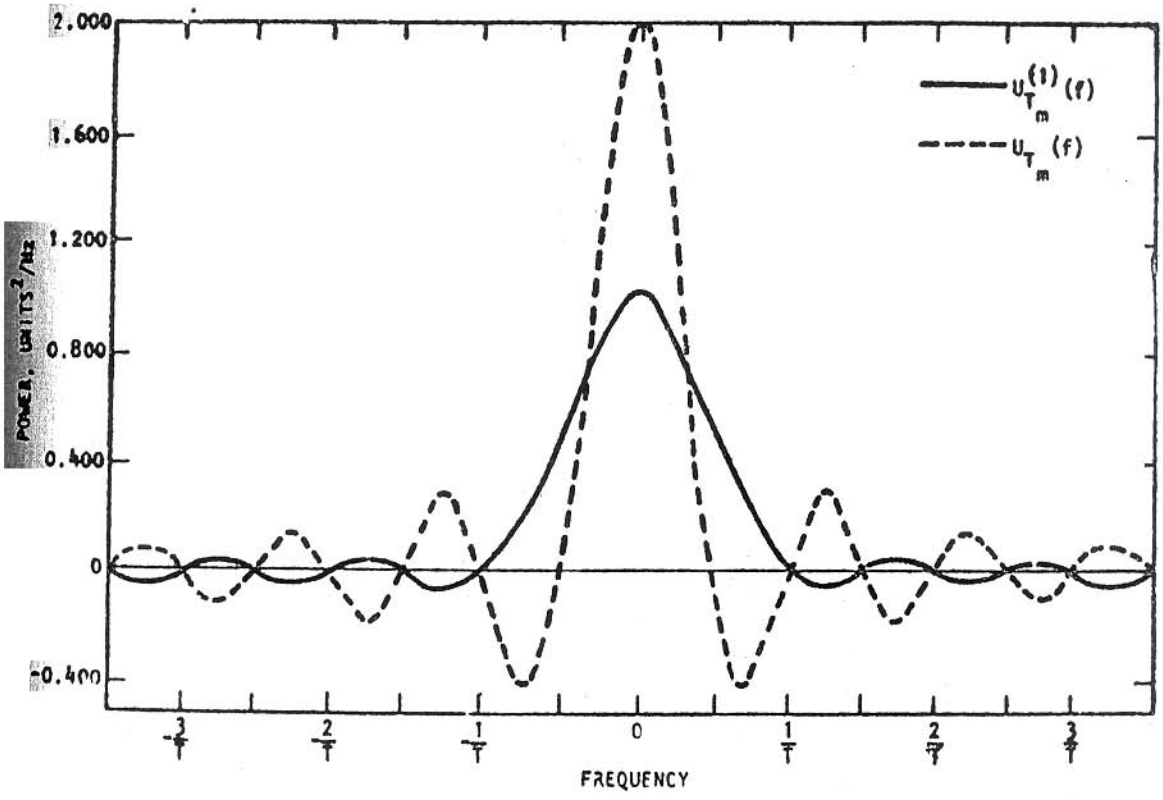
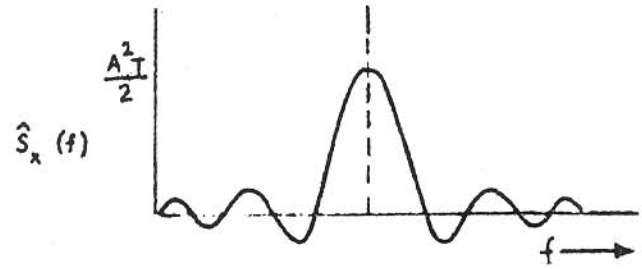


Fig. 2 Power spectral density of a sinusoid (broken line) smoothed by a Hann Window (continuous line). (after Otnes and Enochson, 1972)



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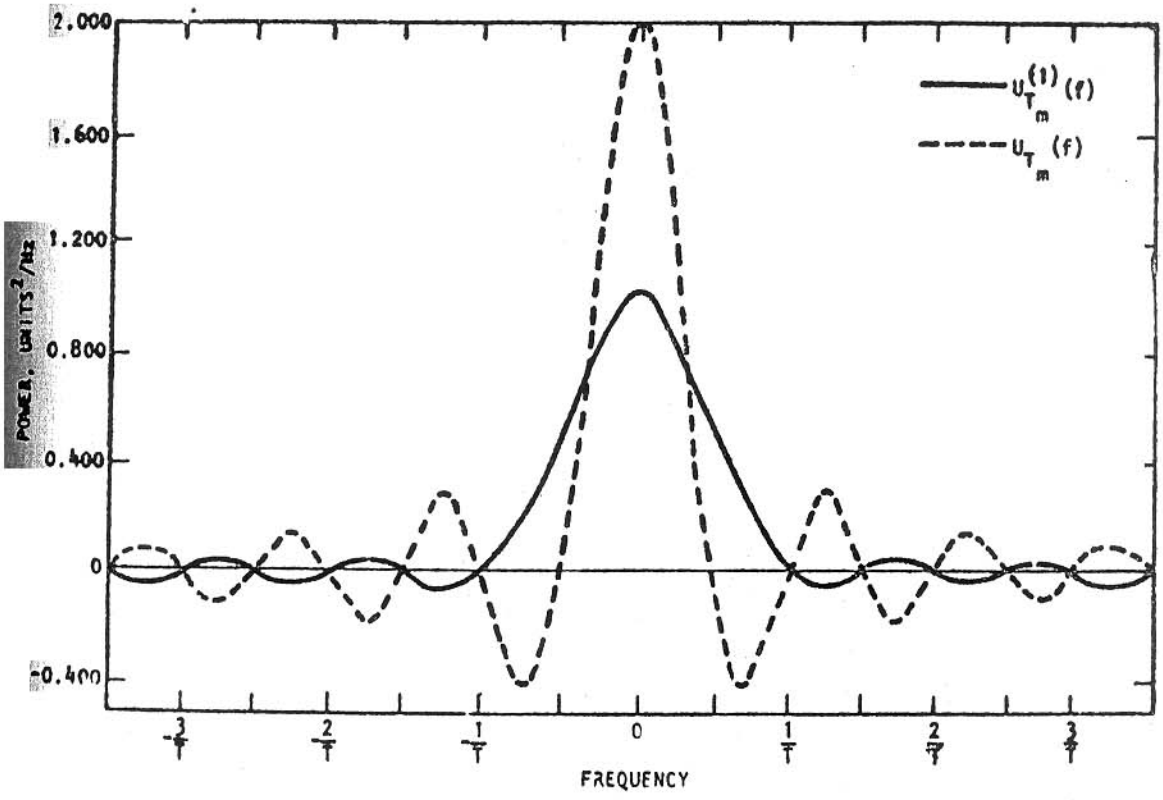

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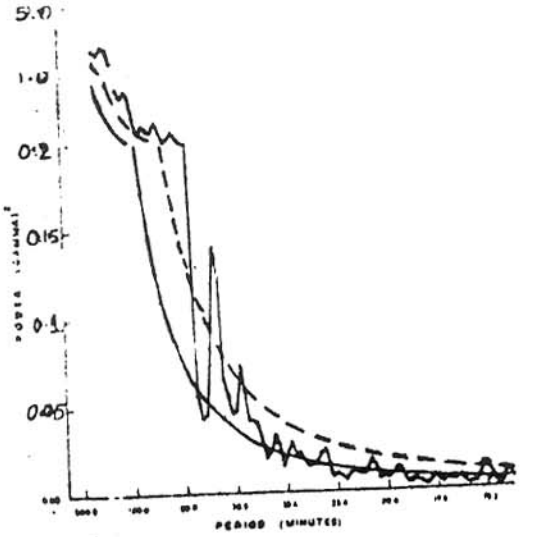

Fig. 3 Power spectrum of horizontal intensity during disturbed conditions. The continuous line is the 'Null' continuum. The broken line representing the 95 percent confidence level for the null continuum.
(after Sen, 1970)

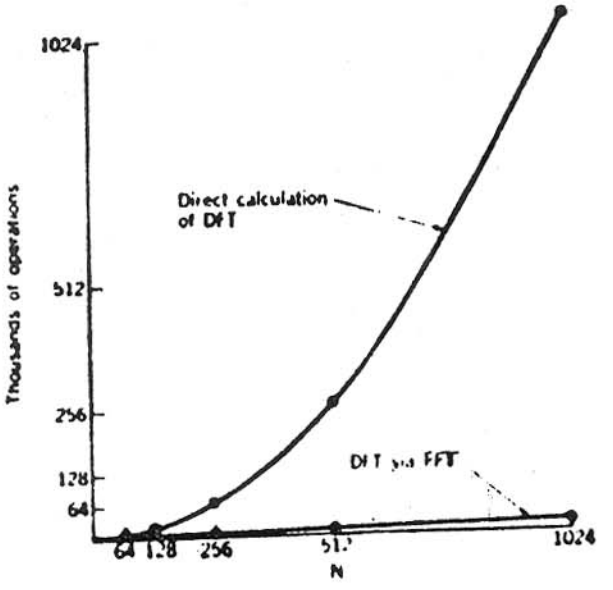

Fig. 4 The number of operations required for computing DFT using FFT algorithm compared with that needed for direct calculation (after Bergland, 1969).

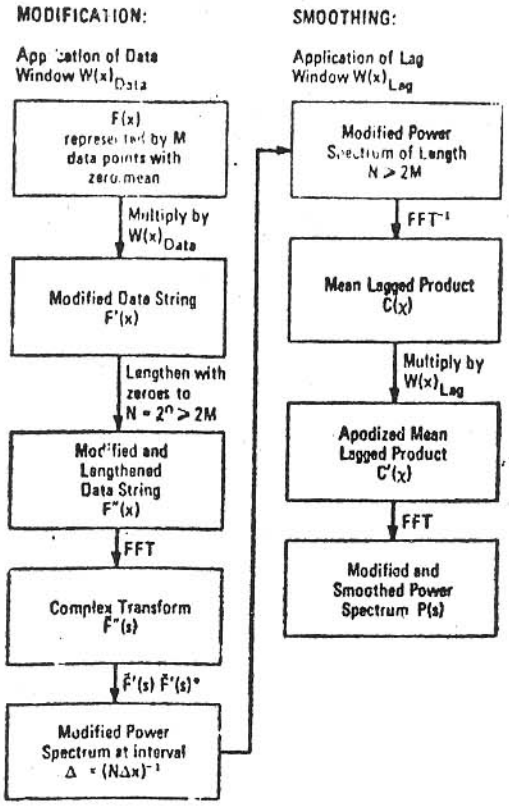

Fig. 5 A flow diagram for spectral analysis of the combined modification and lag window smoothing process using FFT.

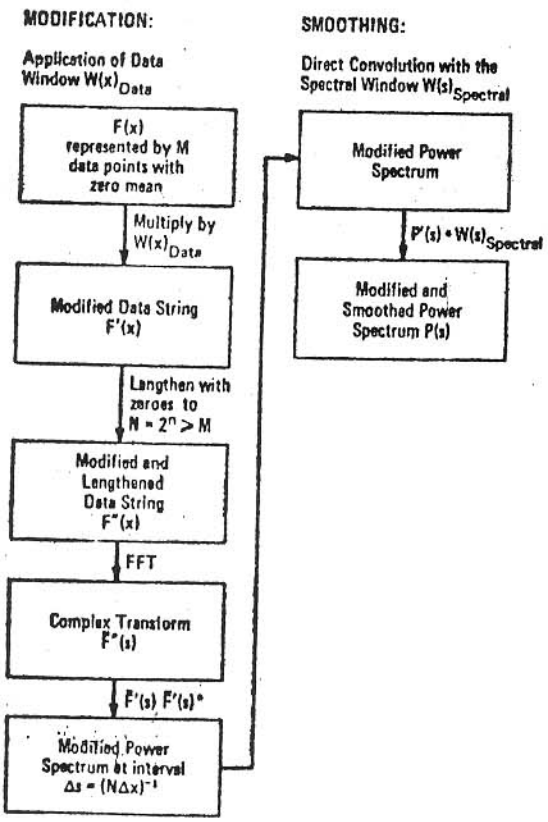

Fig. 6 A flow diagram of the combined modification and direct smoothing method.

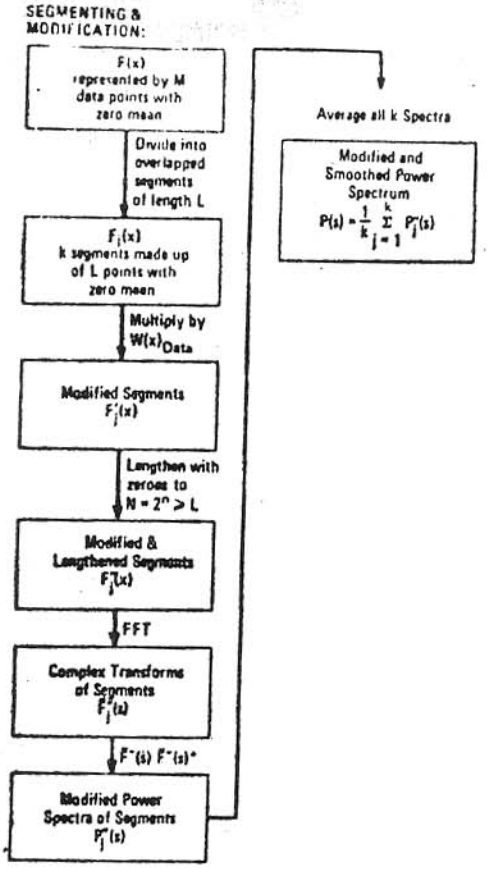

Fig. 7 A flow diagram of the segmental averaging technique for smoothing power spectra using FFT. (Figs. 5, 6 and 7 after Brault and White, 1971).

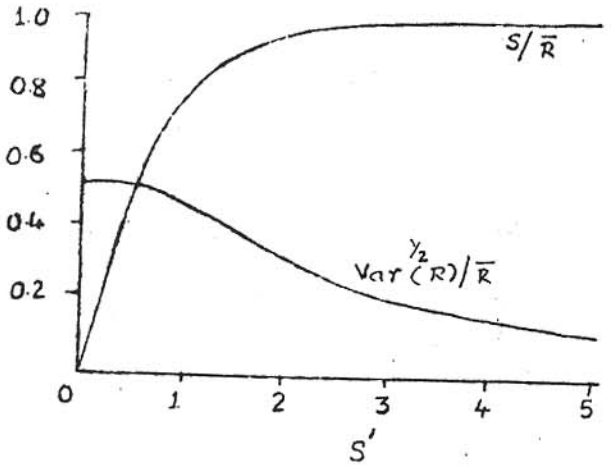

Fig. 8 Curves of Var $^{y2}(R)/\bar{R}$ and $S/\bar{R}$ where R is the "Signal + Noise" amplitude and S is the signal amplitude. These are useful in isolating "noise" contribution in the measured amplitudes (after Black, 1970).

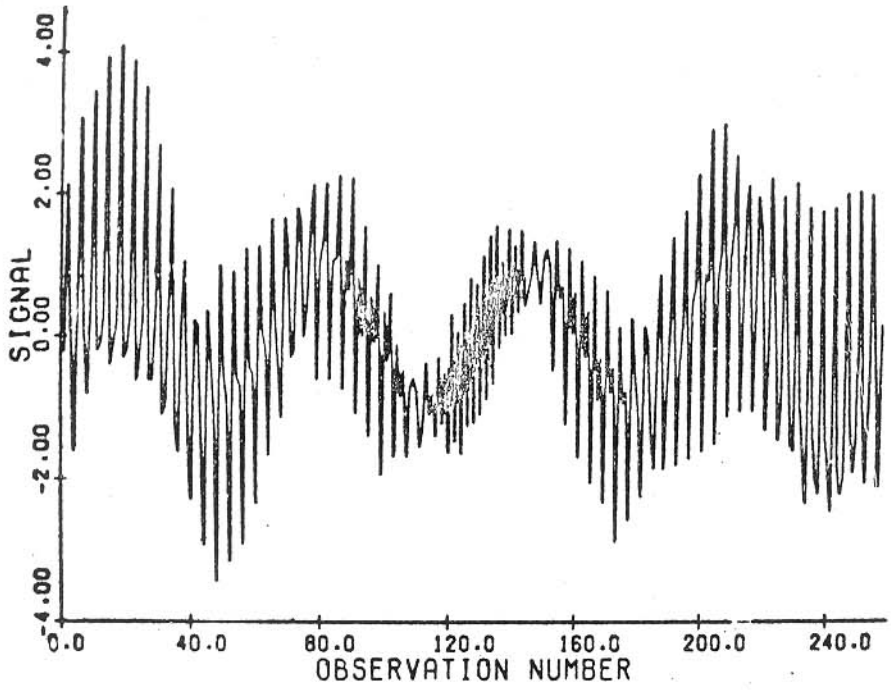

Fig. 9 Trace of the ideal signal sampled at 257 equally spaced points consisting of four equi-amplitude sine wave plus a small amount of white noise. The four waves were such that, if the time represented by the sampled signal shown is 1 sec, then they have frequencies 4, 64, 65 and 124.5 Hz.

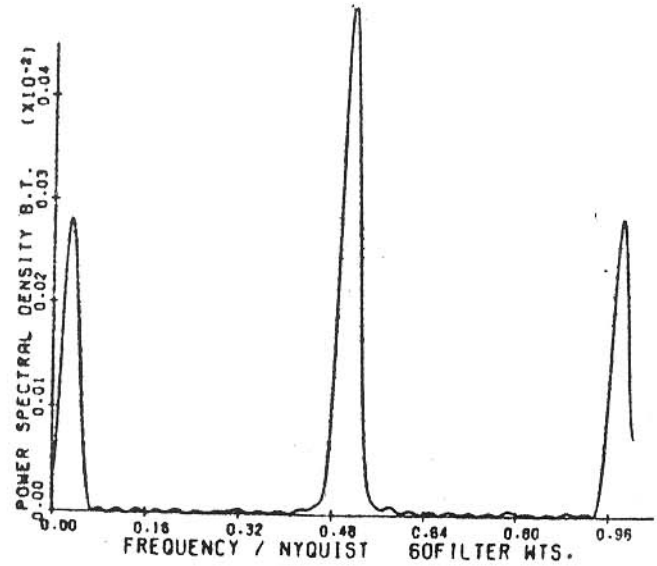

Fig.10 Blackman-Tukey spectrum for the case of 60 lags. It may be seen that the central doublet is still not resolved. To resolve the central doublet one has to go upto 98% lag case (251 lags for 257 data points which is highly unstable.)

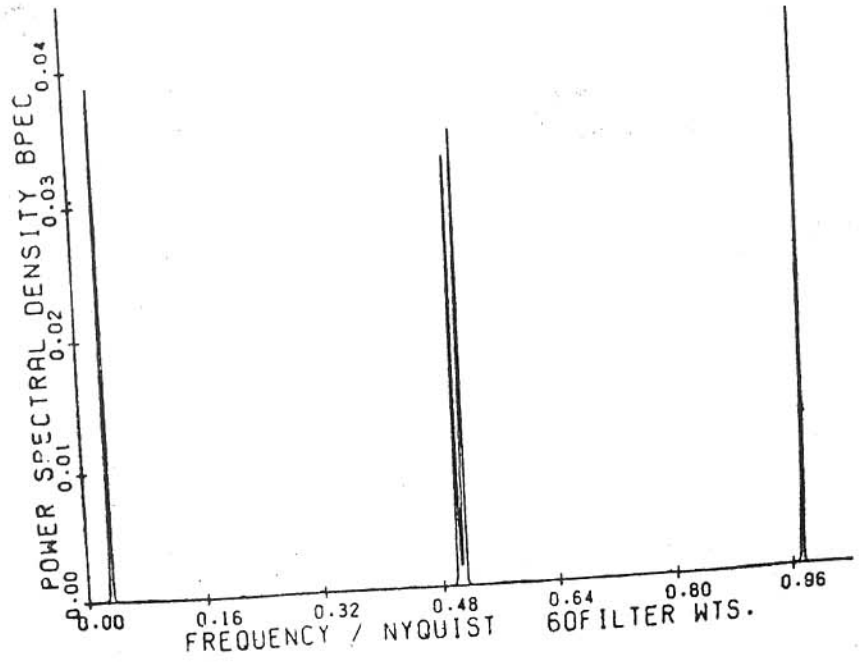

Fig.11 MEM spectrum using 60 filter coefficients. The central doublet is well resolved into its two components. This clearly reveals the superiority of MEM in resolution.
(Figs. 9, 10 and 11 after Radoski et al., 1974)