Oblique non-neutral solitary Alfvén modes in weakly nonlinear pair plasmas

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Abstract. The equal charge-to-mass ratio for both species in pair plasmas induces a decoupling of the linear eigenmodes between waves that are charge neutral or non-neutral, also at oblique propagation with respect to a static magnetic field. While the charge-neutral linear modes have been studied in greater detail, including their weakly and strongly nonlinear counterparts, the non-neutral mode has received less attention. Here the nonlinear evolution of a solitary non-neutral mode at oblique propagation is investigated in an electron–positron plasma. Employing the framework of reductive perturbation analysis, a modified Korteweg–de Vries equation (with cubic nonlinearity) for the lowest-order wave magnetic field is obtained. In the linear approximation, the non-neutral mode has its magnetic component orthogonal to the plane spanned by the directions of wave propagation and of the static magnetic field. The linear polarization is not maintained at higher orders. The results may be relevant to the microstructure in pulsar radiation or to the subpulses.

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Contents

1. Introduction

There is a keen interest in studying large amplitude plasma waves in pair plasmas, such as electron–positron plasmas in pulsar magnetospheres [1]–[9]. The nonlinear interaction of Alfvén waves in electron–positron plasmas is relevant for understanding the pulsar radio emission. Recently, pair plasmas have also been created in the laboratory using charged fullerenes, consisting of C_{60}^+ and C_{60}^- molecules in equal numbers [10]. One could thus, in principle, study the long-term behaviour of nonlinear modes, as one need not worry about the annihilation problems encountered in (laboratory or space) electron–positron plasmas. In pair plasmas, the wave characteristics are quite different from those of electron–ion plasmas, since the natural separation of time and spatial scales associated with the fast and slow motions no longer exists. For example, the ion-acoustic and lower-hybrid type modes cannot exist in pair plasmas.

Stenflo *et al* [6] derived a set of nonlinear equations for the coupling of electromagnetic modes with cold electrostatic waves. Yu *et al* [11] and Bharuthram *et al* [12] investigated, respectively, Alfvén vortices and shear Alfvén wave instabilities caused by current gradients in a magnetized electron–positron plasma. Zhao *et al* [13] were interested in the generation of Alfvén modes by an electron beam in a non-relativistic electron–positron plasma using 3D particle simulation. Their results showed Alfvén waves propagating along the beam as damped solitons, accelerating electrons. Verheest and Lakhina [8] and Lakhina and Verheest [9] studied obliquely propagating solitary Alfvén modes in relativistic electron–positron plasmas. Recently, Verheest and Cattaert [14] gave the arbitrary amplitude generalization of obliquely propagating electromagnetic solitons in pair plasmas, while Schamel and Luque [15] dealt with periodic electrostatic holes and double layers in such plasmas through a kinetic description.

In several of these treatments, the modes studied were charge-neutral, i.e., there were no charge separation effects associated with these modes. For propagation strictly parallel to the external magnetic field, the Alfvén modes in pair plasmas are shown to be charge-neutral in the linear description (see e.g. [16]). Such charge-neutrality also follows from the reductive perturbation analysis [1, 17] for moderately nonlinear modes, and has recently been proved for large solitary structures [14].

For obliquely propagating modes, the picture is more complicated. As we will briefly recall in section 2, there are then two types of linear waves: one charge-neutral, with a dispersion law independent of the angle of propagation (see (7)), whereas the other is non-neutral and its dispersion depends on the angle of propagation (see (8)). The former mode has been studied in the

weakly nonlinear regime by reductive perturbation analysis, and the algebra shows indeed that it conserves its charge-neutral character [8, 9]. This was the argument for Verheest and Cattaert [14] to look for large solitary structures that were assumed to be charge-neutral, in line with the indications of the weakly nonlinear description.

In the present paper, we study nonlinear Alfvén modes propagating at an arbitrary angle to the external magnetic field in a pair plasma by employing a reductive perturbation analysis, in which we retain the displacement current and do not impose a particular polarization for the electromagnetic field waves. Our paper differs from previous treatments in that we, in particular, address the modes which are non-neutral, since, as explained, the charge-neutral modes have been well studied, both for weakly and larger nonlinear amplitudes. To achieve our aim, we let the reductive perturbation formalism decide how the dependent variables are to be expanded order by order, but at strictly oblique propagation.

The paper is organized as follows: in section 2, we delineate the theoretical model and discuss what linear modes are possible in a magnetized pair plasma at arbitrary angles of propagation compared to the direction of the static field. It turns out that in such plasmas the modes decouple into those that are charge-neutral and those that are non-neutral. It is for the latter that we want to find the nonlinear counterpart of, and this is done in section 3. Although not imposed *a priori*, the polarization turns out to be linear at the lowest level, but the higher-order corrections will deviate from that. The nonlinear evolution is governed by a modified Korteweg–de Vries (KdV) equation, with cubic nonlinearity and which in ordinary (multi-ion) plasmas describes electrostatic modes at critical ion densities, whereas the electromagnetic modes obey a derivative nonlinear Schrödinger (DNLS) equation at parallel and a KdV equation at oblique or perpendicular propagation. Finally, our conclusions are given in section 4.

2. Theoretical model and linear modes

For reasons of mathematical tractability, our model is that of a homogeneous, cold plasma immersed in a uniform magnetic field, and composed of equal numbers of negative ions or electrons and positive ions or positrons. Relativistic and thermal corrections do not qualitatively change the character of the nonlinear evolution equation. We study solitary waves propagating along the x-axis, so that all quantities depend only on x and t. For oblique modes we take $\mathbf{B}_0 = B_0 \mathbf{e}_B$, with $\mathbf{e}_B = \cos \vartheta \mathbf{e}_x + \sin \vartheta \mathbf{e}_z$, the unit vector along the static magnetic field with strength B_0. The basic fluid equations include the continuity equations,

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_{\alpha x}) = 0, \tag{1}$$

together with the equations of motion,

$$\frac{\partial \mathbf{v}_\alpha}{\partial t} + v_{\alpha x} \frac{\partial \mathbf{v}_\alpha}{\partial x} = \pm \frac{e}{m}(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}). \tag{2}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, and the label α characterizes the positive ions or positrons ($\alpha = i$, with $q_i = e$ and upper signs in the equations), and the negative particles or electrons ($\alpha = e$, with charge $q_e = -e$ and lower signs), with densities n_α, velocities $\mathbf{v}_\alpha$ and

masses $m = m_e = m_i$. The description is closed by Maxwell's equations,

$$\mathbf{e}_x \times \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{3}$$

$$c^2 \mathbf{e}_x \times \frac{\partial \mathbf{B}}{\partial x} = \frac{\partial \mathbf{E}}{\partial t} + \frac{e}{\varepsilon_0}(n_i \mathbf{v}_i - n_e \mathbf{v}_e), \tag{4}$$

$$\frac{\partial E_x}{\partial x} = \frac{e}{\varepsilon_0}(n_i - n_e), \tag{5}$$

and $B_x = B_{x0} = B_0 \cos \vartheta$ is constant. Before going to the nonlinear development, we linearize and Fourier transform the relevant equations (1)–(5), which yields the linear wave equation,

$$c^2 k^2(\omega^2 - \Omega^2) E_x \mathbf{e}_x + \omega_p^2 \Omega^2 (\mathbf{E} \cdot \mathbf{e}_B)\mathbf{e}_B + [(\omega^2 - c^2 k^2)(\omega^2 - \Omega^2) - \omega_p^2 \omega^2]\mathbf{E} = \mathbf{0}. \tag{6}$$

The total plasma frequency ω_p is defined through $\omega_p^2 = 2n_0 e^2/\varepsilon_0 m$, with $n_0 = n_{e0} = n_{i0}$ the common equilibrium density, and $\Omega = eB_0/m$ is the gyrofrequency in absolute value, for both species. There is thus, at all angles of wave propagation, a mode for which E_y, the component of $\mathbf{E}$ perpendicular to the plane spanned by $\mathbf{k}$ and $\mathbf{B}_0$, decouples from the other components. It has the dispersion law,

$$\omega^4 - \omega^2(c^2 k^2 + \omega_p^2 + \Omega^2) + c^2 k^2 \Omega^2 = 0. \tag{7}$$

This mode has been studied in detail before and corresponds at parallel propagation to a degenerate case of the circularly polarized waves known from standard plasma theory, and at perpendicular propagation to part of the extraordinary mode in the Allis terminology.

   The other components of $\mathbf{E}$, those in the plane spanned by $\mathbf{k}$ and $\mathbf{B}_0$, obey a more complicated dispersion law,

$$\omega^2(\omega^2 - \omega_p^2 - \Omega^2)(\omega^2 - c^2 k^2 - \omega_p^2) - c^2 k^2 \omega_p^2 \Omega^2 \cos^2 \vartheta^2 = 0. \tag{8}$$

For parallel propagation, there occurs a further decoupling between E_x and E_z, where the mode having $E_z \neq 0$ has the same dispersion law (7) as the one with $E_y \neq 0$, while the mode with nonzero E_x corresponds to the cold plasma remnant of the plasma oscillations, at $\omega = \omega_p$, and includes opposite density variations for positive and negative ions. For perpendicular propagation, the waves also decouple. One of the waves corresponds to the incompressible, linearly polarized ($E_z \neq 0$) ordinary mode, again in the Allis terminology, having the usual dispersion law $\omega^2 = c^2 k^2 + \omega_p^2$, whereas the other one is a fixed frequency pure upper-hybrid mode with $\omega^2 = \omega_p^2 + \Omega^2$ and $E_x \neq 0$, the other remnant of the extraordinary mode.

   In the present paper the mode that we further want to study has dispersion properties given by (8). This has a low-frequency, long-wavelength branch, for which the (linear) phase velocity goes as

$$\frac{\omega}{k} = V - VCk^2 + \cdots. \tag{9}$$

Here $V = V_A \cos\vartheta$, and C will be encountered later as the coefficient of the dispersive term in the nonlinear evolution equation,

$$C = \frac{c^2[\omega_p^4 \cos^2\vartheta + (\omega_p^2 + \Omega^2)^2 \sin^2\vartheta]}{2\omega_p^2(\omega_p^2 + \Omega^2)^2}. \tag{10}$$

Because $V \to 0$ when $\vartheta \to 90°$, this mode does not exist at strictly perpendicular propagation, and at parallel propagation it becomes the same generalized X mode that was studied before. Hence, we will concentrate in the nonlinear discussion on purely oblique propagation, with $0 < \vartheta < 90°$, so we assume that both $B_{x0} = B_0 \cos\vartheta \neq 0$ and $B_{z0} = B_0 \sin\vartheta \neq 0$.

The Alfvén velocity V_A itself is given by the equation,

$$V_A^2 = \frac{c^2\Omega^2}{\omega_p^2 + \Omega^2}, \tag{11}$$

which reduces in the limit $\Omega^2 \ll \omega_p^2$ to the expression,

$$V_A^2 \simeq \frac{c^2\Omega^2}{\omega_p^2} = \frac{B_0^2}{2\mu_0 n_0 m}, \tag{12}$$

familiar from weakly magnetized plasmas, but with a factor 2 to account for the proper density definition in a pair plasma. However, in e.g. pulsars, the magnetic field can be large, and hence modifications can occur in V_A if $\omega_p \sim \Omega$. This is also the reason why we cannot *a priori* neglect the displacement current in Ampère's law (4), nor deviations from charge neutrality in Poisson's law (5), as usually done in low-frequency treatments.

3. Nonlinear evolution

To describe nonlinear electromagnetic waves in the low-frequency limit, we first determine a subsidiary equation by taking the scalar product of the equations of motion (2) with **B**, which yields the equation,

$$\mathbf{B} \cdot \left(\frac{\partial}{\partial t} + v_{\alpha x}\frac{\partial}{\partial x}\right)\mathbf{v}_\alpha = \pm\frac{e}{m}\mathbf{E}\cdot\mathbf{B}. \tag{13}$$

Similarly, we see from (2) that the parallel equations of motion can be solved for the expression,

$$E_x = \tfrac{1}{2}[\mathbf{B}_\perp \times (\mathbf{v}_{e\perp} + \mathbf{v}_{i\perp})]\cdot\mathbf{e}_x + \frac{m}{2e}\frac{\partial}{\partial t}(v_{ix} - v_{ex}) + \frac{m}{2e}\left(v_{ix}\frac{\partial v_{ix}}{\partial x} - v_{ex}\frac{\partial v_{ex}}{\partial x}\right). \tag{14}$$

Since ω determined from (9) has a cubic correction in k, the stretching is of the form,

$$\xi = \varepsilon(x - Vt) \qquad \text{and} \qquad \tau = \varepsilon^3 t. \tag{15}$$

We could have equally well taken $\sqrt{\varepsilon}$ instead of ε, provided concomitant changes are made in the expansions of the dependent variables. To let the basic equations determine how the dependent variables are expanded, we could start for all the variables consistently from the equation,

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots. \tag{16}$$

All equilibrium quantities like f_0 are zero, except for the densities and the components of the static magnetic field, as indicated already. However, (13) to lowest order gives that

$$\mathbf{E}_1 \cdot \mathbf{B}_0 = B_0(E_{x1}\cos\vartheta + E_{z1}\sin\vartheta) = 0, \tag{17}$$

so that not only E_x and E_z need to have the same type of expansion, but the lowest-order wave electric field is orthogonal to $\mathbf{B}_0$ and lies in the plane spanned by $\mathbf{k}$ and $\mathbf{B}_0$. Similarly, to lowest order (14) shows that,

$$E_{x1} = -\tfrac{1}{2}B_0(v_{ey1} + v_{iy1})\sin\vartheta, \tag{18}$$

and together with the y component of Faraday's law (3), we note that B_y and $v_{\alpha y}$ also have a similar expansion as

$$B_y = \varepsilon B_{y1} + \varepsilon^3 B_{y3} + \varepsilon^5 B_{y5} + \cdots, \qquad E_x = \varepsilon E_{x1} + \varepsilon^3 E_{x3} + \varepsilon^5 E_{x5} + \cdots,$$
$$E_z = \varepsilon E_{z1} + \varepsilon^3 E_{z3} + \varepsilon^5 E_{z5} + \cdots, \qquad v_{\alpha y} = \varepsilon v_{\alpha y1} + \varepsilon^3 v_{\alpha y3} + \varepsilon^5 v_{\alpha y5} + \cdots. \tag{19}$$

On the other hand, the continuity equations (1), the y components of the equations of motion (2) and the z component of Faraday's law (3) show that the remaining variables have an expansion going as

$$B_z = B_0\sin\vartheta + \varepsilon^2 B_{z2} + \varepsilon^4 B_{z4} + \cdots, \qquad E_y = \varepsilon^2 E_{y2} + \varepsilon^4 E_{y4} + \cdots,$$
$$n_\alpha = n_0 + \varepsilon^2 n_{\alpha2} + \varepsilon^4 n_{\alpha4} + \cdots, \qquad v_{\alpha x} = \varepsilon^2 v_{\alpha x2} + \varepsilon^4 v_{\alpha x4} + \cdots, \tag{20}$$
$$v_{\alpha z} = \varepsilon^2 v_{\alpha z2} + \varepsilon^4 v_{\alpha z4} + \cdots.$$

One can show (and we have checked) that possible intermediate terms in (19) and (20) either vanish or can be renormalized away. To avoid needless cluttering of the expressions, however, these extra terms have been omitted in our exposition.

Substitution of the stretching (15) and the perturbation expansions (19) and (20) into the basic equations (1)–(5) gives a sequence of equations, upon equating the coefficients of the various powers of ε. Some of the intermediate expressions to order ε are,

$$v_{iy1} = v_{ey1} = -\frac{V_A B_{y1}}{B_0}, \tag{21}$$

showing that there will be no lowest-order current in the y direction, and

$$E_{x1} = V_A B_{y1}\sin\vartheta, \qquad E_{z1} = -V_A B_{y1}\cos\vartheta, \tag{22}$$

corroborating the choice for the expansions.

Just to show that the mode is non-neutral, we give the lowest nonzero contributions to the density changes,

$$\frac{n_{i2}}{n_0} = \frac{\Omega^2 B_{y1}^2}{2(\omega_p^2 + \Omega^2)B_0^2} + \frac{V_A \Omega \sin\vartheta}{\omega_p^2 B_0}\frac{\partial B_{y1}}{\partial\xi}, \qquad \frac{n_{e2}}{n_0} = \frac{\Omega^2 B_{y1}^2}{2(\omega_p^2 + \Omega^2)B_0^2} - \frac{V_A \Omega \sin\vartheta}{\omega_p^2 B_0}\frac{\partial B_{y1}}{\partial\xi}. \tag{23}$$

The nonlinearities also generate the field components that were missing to lowest order, as shown in the equation

$$E_{y2} = V_A B_{z2} \cos \vartheta = -\frac{V_A \omega_p^2 B_{y1}^2 \cot \vartheta}{2(\omega_p^2 + \Omega^2) B_0}. \tag{24}$$

Such contributions do not occur at parallel propagation.

Higher-order components quickly become involved, and although the algebra is straightforward, it has to be done with great care. Proceeding thus to the order ε^4, one deduces a modified KdV (mKdV) equation

$$\frac{1}{V} \frac{\partial B_{y1}}{\partial \tau} + A B_{y1}^2 \frac{\partial B_{y1}}{\partial \xi} + C \frac{\partial^3 B_{y1}}{\partial \xi^3} = 0, \tag{25}$$

where C was introduced already in (10) and the new coefficient A is given as

$$A = \frac{3\omega_p^2 \Omega^2}{4 B_0^2 (\omega_p^2 + \Omega^2)^2}. \tag{26}$$

Such an mKdV equation as the governing evolution equation for non-neutral modes in pair plasmas at oblique propagation is to be contrasted to the KdV equation that obtains for similar oblique modes in a standard hydrogen or even multispecies plasma [18]–[21]. The fact that the mKdV equation is odd in its dependent variable, and hence has positive and negative solutions of the same kind, is an indication of the underlying symmetry between the positive and negative particles which characterizes a pair plasma.

The standard one-soliton solution of (25) typically is

$$B_{y1} = \sqrt{\frac{6U}{AV}} \operatorname{sech} \left[\sqrt{\frac{U}{CV}} (\xi - U\tau) \right]. \tag{27}$$

Given that $V = V_A \cos \vartheta$, the amplitude of the soliton blows up as $\vartheta \to 90°$, while its width goes to zero, but, of course, the evolution equation is not valid at strictly perpendicular propagation. For parallel propagation the mKdV equation and its one-soliton solution tally with earlier results [17], where an earlier misprint is corrected.

For weakly magnetized pair plasmas, in the sense that $\Omega^2 \ll \omega_p^2$, we can approximate the coefficients in the mKdV equation as

$$A \simeq \frac{3\Omega^2}{4 B_0^2 \omega_p^2}, \qquad C \simeq \frac{c^2}{2\omega_p^2}, \tag{28}$$

so that the one-soliton solution becomes

$$\frac{B_{y1}}{B_0} = \frac{2\omega_p}{\Omega} \sqrt{\frac{2U}{V_A \cos \vartheta}} \operatorname{sech} \left[\frac{\omega_p}{c} \sqrt{\frac{2U}{V_A \cos \vartheta}} (\xi - U\tau) \right]. \tag{29}$$

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On the other hand, some pulsars can be strongly magnetized ($\omega_p^2 \ll \Omega^2$), and then the coefficients of the mKdV equation have to be approximated as

$$A \simeq \frac{3\omega_p^2}{4B_0^2\Omega^2}, \qquad C \simeq \frac{c^2 \sin^2 \vartheta}{2\omega_p^2}, \tag{30}$$

with the one-soliton solution

$$\frac{B_{y1}}{B_0} = \frac{2\Omega}{\omega_p}\sqrt{\frac{2U}{c\cos\vartheta}} \operatorname{sech}\left[\frac{\omega_p}{c\sin\vartheta}\sqrt{\frac{2U}{c\cos\vartheta}}(\xi - U\tau)\right]. \tag{31}$$

However, when $V_A \to c$, relativistic effects might come into play, which we have omitted in order to avoid cluttering the picture. We have seen in earlier papers [8, 9, 17] that the relativistic effects change the details of the coefficients in the relevant nonlinear evolution equation, but not its generic form, irrespective of it being the KdV or mKdV equation.

4. Conclusions

In pair plasmas, due to the equal charge-to-mass ratios of the negative and the positive charges, the mixing of different scales leads to substantial modifications of the linear and nonlinear evolution of waves. For low-frequency, long-wavelength and non-neutral Alfvén waves propagating at an angle to the external magnetic field and using a proper and consistent reductive perturbation analysis, we find an mKdV equation. It is, however, not valid at strictly perpendicular propagation, but reduces for parallel propagation to the generalized X modes studied by Verheest and Lakhina [8] and Lakhina and Verheest [9].

The pulsar magnetosphere is expected to be dominated by a relativistic electron–positron plasma, although, in general all the three species, namely, electrons, positrons and ions could be present [22]. Kennel and Pellat [23] have shown that when the ions are driven relativistically by large amplitude waves, electrons and ions make equal contributions to the dispersion relation. Therefore, when the wave energy density greatly exceeds the rest mass energy density, it is justified to neglect the differences in rest mass energy between ions and electrons, and such relativistic plasmas can be simply described as electron–positron plasmas with $m_i = m_e$. Therefore, the nonlinear, obliquely propagating Alfvén solitons studied here could be relevant for some problems of pulsar physics, namely, the microstructure in the pulsar radiation or the subpulses, and the acceleration of charged particles to high energies. It is supposed that the subpulses are due to modulation by low-frequency Alfvén solitons of the high-frequency pulsar radiation (itself probably produced by the synchrotron radiation mechanism involving bunches of electrons/positrons moving along the magnetic field lines). Furthermore, interaction of Alfvén solitons with electrons may give rise to strong diffusion and scattering, leading to heating and acceleration of the latter to cosmic energies.

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