

Nonlinear evolution of Alfvénic wave packets

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Abstract. Alfvén waves are a ubiquitous feature of the solar wind. One approach to studying the evolution of such waves has been to study exact solutions to approximate evolution equations. Here we compare soliton solutions of the Derivative Nonlinear Schrödinger evolution equation (DNLS) to solutions of the compressible MHD equations. We find that the soliton solutions of the DNLS equation are not stable solutions of Hall-MHD—they evolve and dissipate with time. Although such solitons may serve as approximate initial conditions to the Hall-MHD equations, they are not stationary solutions. This may account for the absence of soliton-like wave forms in the free-flowing solar wind.

Introduction

Alfvén wave trains have been observed in the solar wind [Belcher and Davis, 1971], in the vicinity of planetary and interplanetary shocks [Agim *et al.*, 1995] and near comets [Scarf *et al.*, 1986; Goldstein *et al.*, 1990]. The evolution of finite amplitude low frequency waves can be best studied by solving the magnetohydrodynamic (MHD) equations, however, those equations are highly nonlinear and are not amenable to analytic solutions except in very special (or linear) cases. Thus it is difficult to explore efficiently all relevant regions of parameter space. Consequently, for finite but small amplitude MHD waves, a variety of evolution equations have been derived which have the advantage that exact analytic solutions can sometimes be found.

One of the most widely used equations is the Derivative Nonlinear Schrödinger (DNLS) equation [Kennel *et al.*, 1988]. Spangler [1997] pointed out that although the DNLS equation is only formally valid for $\delta B/B < 1$, many nonlinear wave characteristics including wavepacket steepening, shocklet formation, and the evolution of polarization can be addressed by the DNLS equation. Consequently, the DNLS formalism has been ap-

plied to the study of low frequency waves upstream of the Earth's bow shock.

However, for large amplitude waves ($\delta B/B \geq 1$), the approximations leading to the DNLS equation become invalid because the derivation includes terms only up to cubic nonlinearities. In addition, solutions of the DNLS equation are not valid for $\beta \sim 1$, where β is the ratio of thermal to magnetic energy. For $\beta \sim 1$, coupling between Alfvén waves and ion acoustic waves becomes significant. Consequently, one has to use scalings different than those used in the derivation of the DNLS. In deriving the DNLS equation the density is assumed to vary on a slower time scale than the magnetic field. Hada [1993], using different stretchings, showed that a system of coupled equations for the density and magnetic field fluctuations is required to describe the interaction of ion acoustic and Alfvén waves.

Recently, Roychoudhury *et al.* [1997], employing a Painlevé analysis of the equations of Hada [1993], showed that, unlike the DNLS equation, the equations studied by Hada [1993] are not completely integrable. However, under translational invariance, it is possible to get a soliton solution in terms of hyperelliptic functions. In this letter we concentrate on comparing the solutions of the DNLS equation with solutions of the Hall-MHD system. We will address in subsequent work the question of how well some of the more recent extensions of the DNLS formalism succeed in describing such phenomena as the formation of discontinuities (see, for example, Mjølhus and Hada [1997]; Medvedev *et al.* [1997]).

In the solar wind, β , T_e and T_i all vary with heliospheric distance. In particular, β spans the entire range from small to large values. Consequently, depending on location in the heliosphere, use of the DNLS equation to explore the evolution of Alfvénic wave packets can be inadequate and it becomes necessary to solve the full set of MHD equations. A major goal of this letter is to ascertain the extent to which solutions of the DNLS equation represent an adequate approximation to solutions of the MHD equations. To explore this issue, we employ a one-dimensional MHD code including the Hall term (to account for the dispersive effects) to study the temporal evolution of large amplitude Alfvénic solitons and wave packets. Our simulation results are valid for any T_e / T_i ratio, but are restricted to quasi-parallel propagation. Because solitons are exact solutions of the DNLS equation, we have used them as initial conditions to the MHD equations. The DNLS equation is not valid for very large amplitude fluctuations, consequently, we

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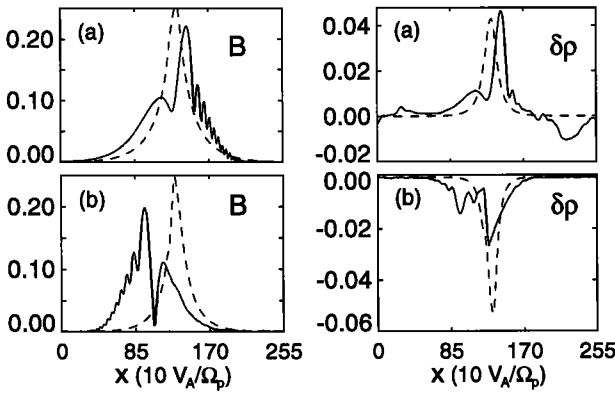


Figure 1. (a) The evolution of the magnetic and density fluctuations, B and $\delta\rho$, respectively, for a RHP soliton for $\beta = 0.3$ at $t \approx 40\Omega_p^{-1}$ (dashed line) and at $5000\Omega_p^{-1}$ (solid line). (b) Same as (a) but for $\beta = 1.5$.

take $\delta B/B = 0.25$ in the MHD solutions shown here. In addition, the solutions we show are for values of β not near unity. The evolution of the solitons is then followed to see if they remain stable.

The evolution of Alfvénic wave packets has been studied previously by *Roberts and Wiltberger* [1995] using a one-dimensional MHD code. That study focused on the evolution of linearly polarized Alfvén waves rather than soliton-like wave packets and concluded that the waves evolved into quasi-steady MHD states that resembled neither solitons nor discontinuities.

Formulation and Simulation Results

Stability of DNLS solitons

For these calculations, we assume that the compressible two-fluid MHD equations include a scalar pressure $p = p(\rho)$, where $p = p_e + p_i$, ρ is the density. In dimensionless units, the equations can be written in the form [*Sakai and Sonnerup*, 1983]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \quad (1)$$

$$\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial}{\partial x}(\rho u_x^2) + \frac{\partial \beta}{\partial x} + \frac{\partial}{\partial x} \left(\frac{b^2}{2} \right) = 0 \quad (2)$$

$$\frac{\partial}{\partial t}(\rho \tilde{v}) + \frac{\partial}{\partial x}(\rho u_x \tilde{v}) - \frac{\partial \tilde{b}}{\partial x} = 0 \quad (3)$$

$$\frac{\partial \tilde{b}}{\partial t} + \frac{\partial}{\partial x}(u_x \tilde{b} - \tilde{v}) = -i \frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial \tilde{b}}{\partial x} \right) \quad (4)$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x}(\beta u_x) + (\gamma - 1)\beta \frac{\partial u_x}{\partial x} = 0 \quad (5)$$

where u_x is the flow velocity along the direction of propagation, b is the magnetic field, $\tilde{b} = (b_y + i b_z)$ and $\tilde{v} = (v_y + i v_z)$. The magnetic fields are normalized to B_0 , velocities to the Alfvén speed, V_A ; pressure to $B_0^2/(4\pi)$; and ρ to ρ_0 . B_0 and ρ_0 are the equilibrium magnetic field and the density respectively.

For $\beta \neq 1$, these equations have been simplified using reductive perturbation methods [*Taniuti and Wei*, 1968], which, for magnetic fluctuations carried to third order, yield the DNLS equation.

$$\frac{\partial B}{\partial t} + \frac{1}{4(1-\beta)} \frac{\partial}{\partial x} (B |B|^2) + \frac{i}{2} \frac{\partial^2 B}{\partial x^2} = 0, \quad (6)$$

where $B = (B_y + i B_z)$. All the quantities in eqn. (6) are dimensionless. Because solutions to the Hall-MHD equations rarely form quasi-stationary final states and almost never relax to simple solitons, we have used exact soliton solutions of the DNLS equation as initial conditions to the MHD equations to test the DNLS solutions. Here we do not address the question of how such a soliton initial condition might arise in nature. We take the following soliton solution of eqn. (6) [*Nozera and Buti*, 1996] as the initial condition for the MHD equations:

$$B(x, t_0) = \frac{(2^{1/2} - 1)^{1/2} B_{max} e^{i\theta(x)}}{[2^{1/2} \cosh(2V_s x) - 1]^{1/2}}, \quad (7)$$

where B_{max} is the amplitude of the soliton,

$$\theta(x) = -V_s x + \frac{3}{8(1-\beta)} \int_{-\infty}^{2x} |B|^2 dx' \quad (8)$$

and V_s is the soliton speed defined by,

$$V_s = \frac{(2^{1/2} - 1) B_{max}^2}{8(1-\beta)}. \quad (9)$$

The MHD system of equations were solved using Fast Fourier Transforms (FFTs) to evaluate the spatial derivatives. For the time integration we used a fourth order Adams-Bashforth scheme [*Hamming*, 1962]. Viscous dissipation (e.g., [*Ghosh et al.*, 1996]) was added to the momentum equation to suppress aliasing errors at high wave numbers.

The initial soliton was defined over a $2550V_A/\Omega_p$ size box with 256 grid points (Ω_p is the proton cyclotron frequency). To check the accuracy of our numerical scheme, we simulated the dynamics of this initial soli-

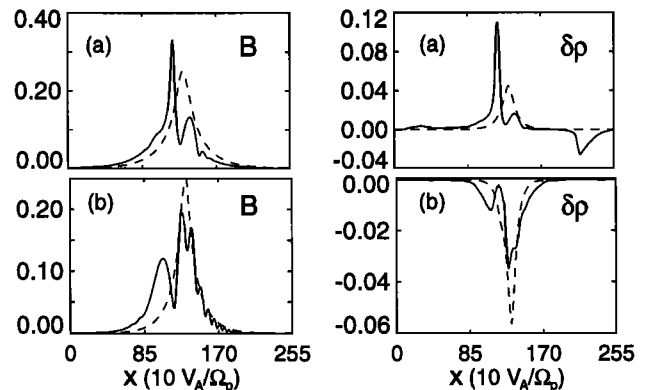


Figure 2. (a) Same as Fig. 1a but for LHP soliton. (b) Same as Fig. 1b but for LHP soliton.

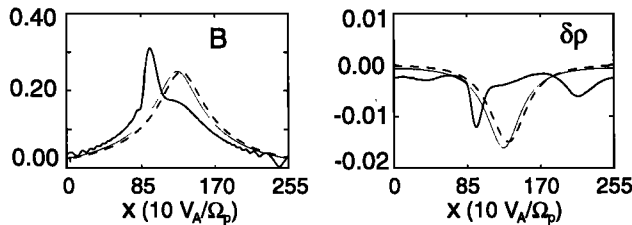


Figure 3. Same as Fig. 1a but for an initial soliton with $\beta = 3.0$. Thicker solid line is for $t = 5000\Omega_p^{-1}$ and thinner one for $t = 0$.

ton using the DNLS equation and confirmed that the soliton remains stable. We first examined the evolution of both right-hand polarized (RHP) and left-hand polarized (LHP) solitons for a range of values of β . The results are shown in Figures 1–3 where the amplitudes of the fluctuations in the magnitude of the magnetic field, B , and fluctuations in the density ρ are plotted. Selected power spectra are plotted in Figure 4.

Figures 1a and 1b illustrate the evolution of the magnetic field and the density from $t = 40\Omega_p^{-1}$ to $t = 5000\Omega_p^{-1}$ for the RHP soliton for $\beta = 0.3$ and $\beta = 1.5$, respectively. The waves steepen at intermediate time, and then form wave trains on the leading edge for $\beta < 1$, and on the trailing edge for $\beta > 1$. The corresponding evolution of LHP solitons with $\beta = 0.3$ and 1.5, respectively are shown in Figures 2a and 2b. Figure 3 shows the evolution for $\beta = 3$ for a RHP soliton. By $t = 5000\Omega_p^{-1}$ (thick solid line) the soliton initial condition has again evolved substantially away from the initial state (thin solid line). In this case however the wave train has almost disappeared. That the soliton initial condition is an approximate equilibrium is emphasized by the fact that at $t = 40\Omega_p^{-1}$ (dashed line) the shape of the wave packet is essentially unchanged from the initial condition.

Figure 4 shows the power spectra of the magnetic field corresponding to the solutions shown in Figures 1 and 2 at $t = 5000\Omega_p^{-1}$. Although all the power spectra

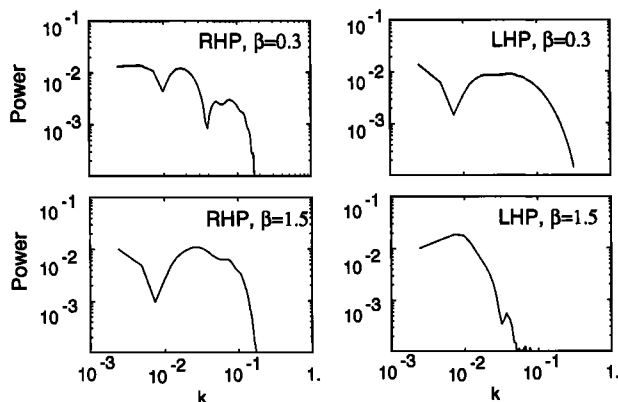


Figure 4. Trace of the wave number power spectra of the fluctuating magnetic field obtained from the wave-form solutions shown in Figures 1 and 2 at time $t = 5000\Omega_p^{-1}$

for each polarization and β differ, none have evolved to having a dominant power law spectrum, at least for the times calculated here. It is clear from Figure 1–3 that the evolution has not progressed to the point that the soliton has become turbulent, nor do we know if it ever will.

The slow variation of the plasma density compared to the transverse magnetic field, which is assumed in deriving the DNLS equation, holds well in the early phase of the nonlinear evolution of the wave packet (cf. $t = 0$ and $t = 40\Omega_p$); however it soon becomes invalid. This failure is independent of both polarization and β . Similar conclusions on the limitations of the DNLS equation were reached by *Spangler* [1997] (also see, *Agim et al.* [1995]). In the DNLS framework, the slow variations in density force a correlation of density with magnetic field—a constraint not imposed on the Hall-MHD equations. Consequently, the relatively smooth and slow evolution of the wave packet indicates that DNLS solitons are approximate equilibrium initial conditions. However, the DNLS solitons are not long-time solutions of the MHD equations.

Conclusions

Solutions of the compressible MHD equations evolve in such a way that the soliton initial condition steepen at the leading or trailing edge depending on the initial wave polarization and also on whether β is less than or greater than unity. On the opposite edges, *i.e.*, on the edges where no steepening is seen, wave trains form as the soliton evolves away from its initial state. The soliton loses its inherent stability even though it is propagating in a homogeneous non-driven system. Such behavior is expected of the DNLS soliton in a driven system [*Hada et al.*, 1990] or in an inhomogeneous system [*Buti*, 1991]. These features indicate that the Alfvén waves are coupling to density fluctuations that are not described properly by the DNLS equation. Because of this coupling, as shown by *Roychoudhury et al.* [1997], the soliton cannot remain coherent.

The DNLS equation relies on the implicit static relationship between magnetic field and density fluctuations and the neglect of higher order couplings (cf. *Spangler* [1997]). These approximations break down with time as seen in the simulation results. Even though the DNLS equation is a good model for studying a number of nonlinear wave properties, the interpretation should be limited to finite time scales, even for relatively small wave amplitudes. One has to rely on the simulation of the Hall-MHD equations to get the true picture of the long term evolution of the wave-trains and wave-packets such as those observed in the solar wind.

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