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Existence domains of slow and fast ion-acoustic solitons in two-ion space plasmas

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A study of large amplitude ion-acoustic solitons is conducted for a model composed of cool and hot ions and cool and hot electrons. Using the Sagdeev pseudo-potential formalism, the scope of earlier studies is extended to consider why upper Mach number limitations arise for slow and fast ion-acoustic solitons. Treating all plasma constituents as adiabatic fluids, slow ion-acoustic solitons are limited in the order of increasing cool ion concentrations by the number densities of the cool, and then the hot ions becoming complex valued, followed by positive and then negative potential double layer regions. Only positive potentials are found for fast ion-acoustic solitons which are limited only by the hot ion number density having to remain real valued. The effect of neglecting as opposed to including inertial effects of the hot electrons is found to induce only minor quantitative changes in the existence regions of slow and fast ion-acoustic solitons. © *2015 AIP Publishing LLC*.
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I. INTRODUCTION

Nonlinear solitary wave structures often manifest themselves as travelling bipolar pulses in the magnetic field-aligned component of electric field data. These waveforms are indicative of positive polarity potential structures that have been observed in the mid-altitude auroral zone,[1] polar magnetosphere,[2] bow shock,[3] plasma sheet boundary layer,[4] and more recently in the dayside magnetosheath.[5] A theoretical interpretation for these nonlinear waveforms was given in terms of Bernstein-Green Kruskal (BGK) modes;[6] however, these could also now be interpreted as electron-acoustic type solitary waves in view of the theoretical models which support high frequency positive polarity structures.[7–12] On the other hand, low frequency signatures of electrostatic solitary wave and double layers observed in ion data by S3-3 and Viking in the auroral regions,[13–15] having negative potentials are most likely nonlinear structures of the ion-acoustic type. There is a vast body of theoretical papers which focus on ion (electron) dynamics in studies on ion (electron)-acoustic solitons and double layers. The characteristics of large and small amplitude ion-acoustic double layers were investigated by Bharuthram and Shukla[16] for a model with cold ions and Boltzmann distributed cool and hot electron populations. The explanation as to why negative potential double layers occur in a model with positive ions became evident in a later study by Baboolal *et al.*,[17] wherein it was found that double layers occur as upper limits on the existence regions of solitons. Only double layers were investigated in an earlier paper by Baboolal *et al.*[18] The studies in Refs. 17 and 18 were based on models with one and two (different mass) species of adiabatic ions. Contrary to the belief that solitons are superacoustic, i.e., they can only propagate at speeds which exceed the sound speed, positive potential solitons were found to propagate at the acoustic speed in Ref. 17. This appeared to have remained unnoticed for many years even by the authors themselves up until very recently when this was pointed out by Baluku *et al.*[19] and formed the basis for a study in which the focus was on parameter regions where ion-acoustic solitons propagate at the acoustic speed. It was also found in Ref. 19 that positive potential double layers are possible which is unexpected for the model with positively charged ions, and Boltzmann cool and hot electrons as one is inclined to believe that positive potential solitons limited by number density considerations of the ions should coexist with negative solitons limited by double layers.[17] The early theoretical studies on high frequency electron-acoustic solitons were based on models which neglected inertial effects of the hot electrons;[20] consequently, only negative potential solitons were found. In later studies, it became evident that positive polarity electron-acoustic solitons are possible by either treating the hot electrons as inertial[9] in the two-electron model with cool and hot electrons; or if the choice is to treat the hot electrons as inertialess (Boltzmann distributed) then the two-electron component model has to be extended to include an additional component of inertial electrons with[21] or without[10] a drift. In view of the findings in Ref. 10, we are of the opinion that it is not so much streaming as much as it is inertial effects of an additional electron component that is required to invoke a switch to positive polarity structures in the supported electron-acoustic solitons. Studies on ion-acoustic solitons with two species of ions[22,23] were found to support only positive polarity structures. The traditional Sagdeev approach was used in the study in Ref. 22, whereas the fluid-dynamic paradigm[24–26] was adopted in Ref. 23. It was found that the

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cool (positive) ions which are supersonic (thermal speed is less than the speed of the nonlinear structure) are responsible for the upper Mach number limitation for solitons found in a model which was also composed of hot (negative) ions which are subsonic (thermal speed is greater than the speed of the nonlinear structure). In the model of Ref. 23 in which only inertia was retained (thermal pressure was neglected) for the cool ions and only thermal pressure was retained (inertia was neglected) for the hot ions and the electrons, it was found that there is a switch from the limit being imposed by the cool ions to that of the hot ions for the value $\gamma = 2$ (polytropic index of the two inertialess species). In the Boltzmann limit ($\gamma = 1$) for the hot electrons and hot ions, the hot ion number density limit was eliminated and only the cool ions were found to be responsible for the upper Mach number limit.

The one-ion model[11] was revisited by Maharaj *et al.*,[27,28] wherein the objectives were to broaden the scope of the findings in Ref. 11 by investigating the upper Mach number limitations for ion-acoustic[27] and electron-acoustic[28] solitons. These studies provided very valuable insights into how the constraint imposed by the ion number density limits ion-acoustic solitons[27] and how number density considerations of the cool and hot electrons, and the occurrence of negative and positive double layers place constraints on the existence regions of electron-acoustic solitons.[28] It has already been established in Ref. 12, that by extending the one-ion plasma model to a model with two-temperature ions, two ion-acoustic modes can be supported, viz., the lower phase speed (or slow) ion-acoustic soliton which is associated with the presence of the cool ions in the model; whereas a higher phase speed or fast ion-acoustic soliton is associated with the presence of the hot ions in the model. Furthermore, we recall from the results presented in Ref. 12 that a polarity switch is possible for electron-acoustic solitons, but only positive polarity structures were found for both slow and fast ion-acoustic solitons. Two distinct ion-acoustic modes, viz., the lower phase speed (slow) and higher phase speed (fast) ion-acoustic modes are supported in plasmas if there is a difference in mass between the two inertial ion components and the temperatures (or gradients in pressures) should also be different. The properties of slow and fast ion-acoustic solitons are discussed in a number of theoretical studies[32–34] and in the context of laboratory experiments.[35,36] In Ref. 37, a range (band) of velocities over which fast ion-acoustic solitons cannot propagate was found.

Here, we will revisit the two-ion model of Lakhina *et al.*,[12] composed of cool and hot ions, and cool and hot electrons which aptly models the plasma composition in the plasma sheet boundary layer (PSBL) region.[29] Magnetospheric plasmas which are composed of cooler, heavier ions (usually positively charged oxygen) and hotter, lighter ions (protons)[38] are quite common. In the PSBL region of the magnetosphere, the hot ions are much hotter[29] than the cool electrons and hence this means that there is a not a very wide separation between the hot ion and cool electron thermal speeds. The objective of this study is to model a situation where all species including the electrons have a sluggish response. Since the cool and hot electrons species are treated as adiabatic fluids rather than isothermal and inertialess (Boltzmann) species, the model reinforces the idea that on timescales of the ions, the cool and hot electrons will not have sufficient time to thermalise to a single electron species plasma. Thermalization of the electrons could be easily achieved if once considers very long timescales associated with dust dynamics, then, it makes sense that the electrons in a dusty plasma model should be treated as isothermal (Boltzmann distributed). It will be seen later that the phase speeds of the slow and fast ion-acoustic modes differ at most by one order of magnitude; hence, the slow ion-acoustic mode phase speed is not low enough to warrant thermalisation of the electrons to a single temperature. In this study, the hot ions will not be considered hotter than the hot electrons as this will result in strong Landau damping of the linear ion-acoustic waves. In our study, we will assume that the hot ions and cool electrons have the same temperature but the hot electrons are hotter than the hot ions. This will also allow for broader applicability of the theoretical model to other regions in the Earth's magnetosphere than being restricted to the PSBL[29] region but also provides sufficient justification for the model where all species can be treated as adiabatic fluids.

We will extend here the scope of the study in Ref. 12 by considering why upper Mach number limitations occur for slow and fast ion-acoustic solitons. In view of the findings in Refs. 27 and 28, it is clear that in seeking existence regions for solitons, the merits of considering why upper Mach number limitations arise for solitons by far outweighs the benefits of attempting to identify soliton existence regions by plotting Sagdeev potentials for different combinations of fixed parameters as interesting soliton existence regions such as those where polarity switches occur could easily be missed. In addition, although this was not considered in Ref. 12 here we will also establish to what extent is the Boltzmann assumption for the hot electrons a good approximation by comparing the existence regions obtained from assuming an adiabatic fluid response for the hot electrons as opposed to treating the hot electrons as being Boltzmann distributed.

The outline of the paper is as follows. In Sec. II, we present details of the theory for the four-component model composed of cool ions, hot ions, cool electrons, and hot electrons, for which inertia and pressure are included for all species.[12] The model with Boltzmann hot electrons is discussed in Sec. III. Existence regions of large amplitude slow and fast ion-acoustic solitons are presented and discussed in Sec. IV. A summary of our findings and conclusions are given in Sec. V.

II. GOVERNING EQUATIONS FOR A MODEL WHICH INCLUDES INERTIAL EFFECTS OF THE HOT ELECTRONS

We consider an unmagnetized plasma composed of cool ions, hot ions, cool electrons, and hot electrons. Including inertia and pressure for all four plasma species, the continuity, momentum, and pressure equations for all four species are given by

$$\frac{\partial n_j}{\partial t} + \frac{\partial (n_j v_j)}{\partial x} = 0, \quad (1)$$

$$\frac{\partial v_j}{\partial t} + v_j\frac{\partial v_j}{\partial x} = -\frac{Z_j}{\mu_j}\frac{\partial \Phi}{\partial x} - \frac{1}{\mu_j n_j}\frac{\partial P_j}{\partial x}, \tag{2}$$

$$\frac{\partial P_j}{\partial t} + v_j\frac{\partial P_j}{\partial x} + 3P_j\frac{\partial v_j}{\partial x} = 0, \tag{3}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = n_{ce} + n_{he} - n_{ci} - n_{hi}, \tag{4}$$

where n_j, v_j, T_j, and $P_j = n_jT_j$ denotes, respectively, the normalized number density, fluid velocity, temperature, and pressure of species j, where $j = ci, hi, ce$, and he, respectively, denotes the cool ions, hot ions, cool electrons, and hot electrons. Furthermore, Φ is the normalized wave potential, $\mu_{ce} = \mu_{he} = \mu_e = m_e/m_i$, where m_j denotes the mass of species j, $\mu_{ci} = \mu_{hi} = 1$, $Z_{ce} = Z_{he} = -1$ for the cool (or hot) electrons and $Z_{ci} = Z_{hi} = 1$. All densities are normalized with respect to the total equilibrium electron (or ion) number density, viz., $n_0 = n_{ci0} + n_{hi0} = n_{ce0} + n_{he0}$, velocities are normalized with respect to the hot ion thermal speed $C_{hi} = (T_{hi}/m_i)^{1/2}$, time with respect to the inverse ion plasma frequency $\omega_{pi}{}^{-1} = (m_i/4\pi n_0 e^2)^{1/2}$, lengths with respect to the ion Debye length $\lambda_{di} = (T_{hi}/4\pi n_0 e^2)^{1/2}$, potential with respect to T_{hi}/e, and thermal pressures with respect to $n_0 T_{hi}$. For all species, we have assumed an adiabatic fluid response and used the same value for the polytropic index, viz., $\gamma = 3$.

We transform the set of Eqs. (1)–(4) to a frame moving with the wave through the co-moving coordinate $\xi = x - Mt$, where the Mach number, $M(=V/C_{hi})$ denotes the speed of the nonlinear structures, normalized with respect to the hot ion thermal speed. Following the mathematical procedure in Mendoza-Briceño *et al.*,[30] we solve for the densities for the different species by making use of the boundary conditions which are given by

$$\Phi \to 0,\ \mathrm{d}\Phi/\mathrm{d}\xi \to 0,\ n_{ci} \to n_{ci}^0,\ n_{hi} \to n_{hi}^0,\ n_{ce} \to n_{ce}^0,$$
$$n_{he} \to n_{he}^0,\ P_{ci} \to n_{ci}^0 T_{ci},\ P_{hi} \to n_{hi}^0,\ P_{ce} \to n_{ce}^0 T_{ce},$$
$$P_{he} \to n_{he}^0 T_{he},\ \text{as } \xi \to \pm\infty, \tag{5}$$

where $n_j^0 = n_{j0}/n_0$ such that $n_{ce}^0 + n_{he}^0 = n_{ci}^0 + n_{hi}^0 = 1$. Solving for the number densities and expressing these in the form $n_j = n_{j0}(\sqrt{a} \pm \sqrt{b})$ as in Ghosh *et al.*,[31] respectively, yields for the cool and hot ions, cool and hot electrons the expressions given by

$$n_{ci} = \frac{n_{ci}^0}{2\sqrt{3T_{ci}}}\left\{\left[\left(M + \sqrt{3T_{ci}}\right)^2 - 2\Phi\right]^{1/2} \pm \left[\left(M - \sqrt{3T_{ci}}\right)^2 - 2\Phi\right]^{1/2}\right\}, \tag{6}$$

$$n_{hi} = \frac{n_{hi}^0}{2\sqrt{3}}\left\{\left[\left(M + \sqrt{3}\right)^2 - 2\Phi\right]^{1/2} \pm \left[\left(M - \sqrt{3}\right)^2 - 2\Phi\right]^{1/2}\right\}, \tag{7}$$

$$n_{ce} = \frac{n_{ce}^0}{2\sqrt{3T_{ce}/\mu_e}}\left\{\left[\left(M + \sqrt{3T_{ce}/\mu_e}\right)^2 + (2\Phi/\mu_e)\right]^{1/2} \pm \left[\left(M - \sqrt{3T_{ce}/\mu_e}\right)^2 + (2\Phi/\mu_e)\right]^{1/2}\right\}, \tag{8}$$

$$n_{he} = \frac{n_{he}^0}{2\sqrt{3T_{he}/\mu_e}}\left\{\left[\left(M + \sqrt{3T_{he}/\mu_e}\right)^2 + (2\Phi/\mu_e)\right]^{1/2} \pm \left[\left(M - \sqrt{3T_{he}/\mu_e}\right)^2 + (2\Phi/\mu_e)\right]^{1/2}\right\}. \tag{9}$$

The choice of the lower sign (minus) in the density expressions given by Eqs. (6)–(9) is consistent with the boundary conditions (Eq. (5)). On substituting these in Poisson's equation and integrating it yields the energy integral-like equation

$$\frac{1}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right)^2 + V(\Phi, M) = 0, \tag{10}$$

where our expression for the Sagdeev potential reads as

$$\begin{aligned} V(\Phi, M) = {} & \frac{n_{hi}^0}{6\sqrt{3}}\left\{\left(M + \sqrt{3}\right)^3 - \left(\sqrt{\left(M + \sqrt{3}\right)^2 - 2\Phi}\right)^3\right\} - \frac{n_{hi}^0}{6\sqrt{3}}\left\{\left(M - \sqrt{3}\right)^3 - \left(\sqrt{\left(M - \sqrt{3}\right)^2 - 2\Phi}\right)^3\right\} \\ & + \frac{n_{ci}^0}{6\sqrt{3T_{ci}}}\left\{\left(M + \sqrt{3T_{ci}}\right)^3 - \left(\sqrt{\left(M + \sqrt{3T_{ci}}\right)^2 - 2\Phi}\right)^3\right\} \\ & - \frac{n_{ci}^0}{6\sqrt{3T_{ci}}}\left\{\left(M - \sqrt{3T_{ci}}\right)^3 - \left(\sqrt{\left(M - \sqrt{3T_{ci}}\right)^2 - 2\Phi}\right)^3\right\} \end{aligned}$$

$$+\frac{n_{ce}^{0}\mu_e}{6\sqrt{3T_{ce}/\mu_e}}\left\{\left(M+\sqrt{3T_{ce}/\mu_e}\right)^3-\left(\sqrt{\left(M+\sqrt{3T_{ce}/\mu_e}\right)^2+\frac{2\Phi}{\mu_e}}\right)^3\right\}$$
$$-\frac{n_{ce}^{0}\mu_e}{6\sqrt{3T_{ce}/\mu_e}}\left\{\left(M-\sqrt{3T_{ce}/\mu_e}\right)^3-\left(\sqrt{\left(M-\sqrt{3T_{ce}/\mu_e}\right)^2+\frac{2\Phi}{\mu_e}}\right)^3\right\}$$
$$+\frac{n_{he}^{0}\mu_e}{6\sqrt{3T_{he}/\mu_e}}\left\{\left(M+\sqrt{3T_{he}/\mu_e}\right)^3-\left(\sqrt{\left(M+\sqrt{3T_{he}/\mu_e}\right)^2+\frac{2\Phi}{\mu_e}}\right)^3\right\}$$
$$-\frac{n_{he}^{0}\mu_e}{6\sqrt{3T_{he}/\mu_e}}\left\{\left(M-\sqrt{3T_{he}/\mu_e}\right)^3-\left(\sqrt{\left(M-\sqrt{3T_{he}/\mu_e}\right)^2+\frac{2\Phi}{\mu_e}}\right)^3\right\}. \tag{11}$$

The requirements for (10) to yield soliton solutions are: (i) $V(\Phi) = dV(\Phi)/d\Phi = 0$ at $\Phi = 0$, (ii) $(d^2V(\Phi)/d\Phi^2)_{\Phi=0} < 0$ (the origin is an unstable fixed point), (iii) $V(\Phi) = 0$ at $\Phi = \Phi_{n(p)}$ where $\Phi_{n(p)}$ is a negative (positive) root of $V(\Phi) = 0$ such that $V(\Phi) < 0$ for $\Phi_n < \Phi < 0$ for negative potential solitons and $0 < \Phi < \Phi_p$, for positive potential solitons, (iv) $(d^3V(\Phi)/d\Phi^3)_{\Phi=0} < 0$ for negative potential solitons and $(d^3V(\Phi)/d\Phi^3)_{\Phi=0} > 0$ for positive potential solitons. In addition, a negative (positive) potential soliton solution requires that (v) $(dV(\Phi)/d\Phi)_{\Phi=\Phi_n} < 0$ for negative potential solitons and $(dV(\Phi)/d\Phi)_{\Phi=\Phi_p} > 0$ for positive potential solitons ensuring that a pseudo particle experiences a force in the direction of increasing negative (decreasing positive) values of Φ for negative (positive) potential solitons so that it is reflected back to the origin ($\Phi = 0$).

The requirements for a double layer solution is that in addition to the conditions (i), (ii), and (iii) for solitons, the requirement (vi) $(dV(\Phi)/d\Phi) = 0$ must be satisfied at $\Phi = \Phi_n$ (or $\Phi = \Phi_p$) for a negative (positive) double layer.

It is important to point out here that small amplitude soliton solutions which are obtained from the Korteweg-de Vries (KdV) equation should be in reasonably good agreement with the soliton results found using the arbitrary amplitude theory for soliton amplitudes which are not too large. One must, however, still bear in mind that there are significant differences in the results obtained from the two approaches in that contrary to solitons obtained from the arbitrary amplitude approach, KdV solitons cannot propagate at the acoustic speed and coexistence (of opposite polarity) KdV solitons is not supported. The polarity of a KdV soliton is dictated by the sign of the coefficient of nonlinear term in the KdV equation which is the sign of the third derivative of the Sagdeev potential (at $\Phi = 0$) stipulated as the condition (iv) mentioned earlier. In the limit of small amplitude, one may Taylor expand $V(\Phi)$ to third order to obtain

$$V(\Phi) \approx C_2\Phi^2 + C_3\Phi^3, \tag{12}$$

for which the solution for KdV solitons can be written as

$$\Phi = -\left(\frac{C_2}{C_3}\right)\mathrm{sech}^2\left(\sqrt{-\frac{C_2}{4}\xi^2}\right), \tag{13}$$

where

$$C_2 = \frac{1}{2}\left(\frac{d^2V(\Phi)}{d\Phi^2}\right)_{\Phi=0} = \frac{n_{ci}^0}{[M^2-3T_{ci}]} + \frac{n_{hi}^0}{[M^2-3]} + \frac{n_{ce}^0}{\mu_e[M^2-(3T_{ce}/\mu_e)]} + \frac{n_{he}^0}{\mu_e[M^2-(3T_{he}/\mu_e)]} \tag{14}$$

and

$$C_3 = \frac{1}{6}\left(\frac{d^3V(\Phi)}{d\Phi^3}\right)_{\Phi=0} = \frac{n_{ci}^0[M^2+T_{ci}]}{2[M^2-3T_{ci}]^3} + \frac{n_{hi}^0[M^2+1]}{2[M^2-3]^3} - \frac{n_{ce}^0[M^2+(T_{ce}/\mu_e)]}{2\mu_e^2[M^2-(3T_{ce}/\mu_e)]^3} - \frac{n_{he}^0[M^2+(T_{he}/\mu_e)]}{2\mu_e^2[M^2-(3T_{he}/\mu_e)]^3}. \tag{15}$$

The expression (14) is identical with that given as (7) in Ref. 12 assuming that none of the species are drifting. The root(s) which correspond to the vanishing of (14) are the phase speeds of the linear wave modes supported by the model. These are the critical values of the Mach number, M_{crit}, which are the minimum allowed values of the Mach number for which soliton solutions can be obtained for the particular plasma composition. In order to determine the critical values of M, we solve (14) numerically. For the parameter regions investigated in this paper, three distinct positive roots corresponding to the vanishing of C_2 have been obtained of which the smallest value of M_{crit} was identified in Ref. 12 as a slow ion-acoustic mode, the intermediate root is a fast ion-acoustic mode and the largest root is an electron-acoustic mode. These critical values of the Mach number provide lower limits on the velocity ranges of solitons associated with the three different linear wave modes. The distinction between cool and hot ions in the theoretical model is what invokes the existence of slow and fast ion-acoustic waves, hence the existence of the former is ruled out[34] when there are no cool ions in the plasma composition. The sign of $C_3(M)$ evaluated at $M = M_{\mathrm{crit}}$ dictates what the polarity of the solitons should be in the limit of small amplitude as stated earlier by the condition (iv). The expression for C_3 is very useful even in studies on solitons of arbitrary amplitude since polarity switches in large amplitude solitons occur precisely at the point where $C_3 = 0$ between regions in parameter space where positive ($C_3 > 0$) and negative ($C_3 < 0$) potential KdV solitons are supported.

Having mentioned already that a minimum value of the Mach number is required for solitons to be possible, it is important to point out that there also exist upper Mach number limitations for solitons, consequently establishing that there is a limit on the amplitude of the structures. Considering simple two-component plasma models with inertial positive ions in addition to the electrons which can be treated either as inertial or inertialess and Boltzmann distributed, it is known that only positive polarity ion-acoustic solitons will be supported, which is consistent with the sign of the inertial ions in the model. The amplitude of the positive potential solitons will be limited by the constraint that the number density of the ions should not become complex valued. Extending the one-ion model to a two-ion model with inertial cool and hot ions (both positively charged), will not only invoke the existence of both slow and fast ion-acoustic solitons but this also now imposes two possibilities as to why there is an upper limit on the amplitude, and consequently, the Mach number of the positive potential soliton structures; the limit on the amplitude of the positive polarity soliton structures could either be imposed by the number density of the cool ions (6) or the hot ions (7) becoming complex valued. If the cool ions are responsible for the upper limit on M, then the maximum permitted value of the positive potential, viz., $\Phi_{\mathrm{max/cool}} = (M - \sqrt{3T_{ci}})^2/2$ such that the cool ion number density (6) remains real valued establishes the existence of a limit on the amplitude of the positive polarity solitons. The amplitude of the solitons will increase with increasing Mach number only up until Φ exceeds $\Phi_{\mathrm{max/cool}}$, then the cool ion number density becomes complex valued and solitons will no longer occur. The upper limiting value of M in this case is precisely that value of M which coincides with the limit $\Phi = \Phi_{\mathrm{max/cool}}$. Similarly, the other possibility is that the hot ions in the model could be responsible for the limit on the amplitude of the positive potential solitons, in which case, Φ should not exceed $\Phi_{\mathrm{max/hot}} = (M - \sqrt{3})^2/2$, else, the hot ion number density (7) becomes complex valued and positive potential solitons will no longer occur. When the hot ions are responsible for limiting the amplitudes of the positive polarity solitons, the upper M limit coincides precisely with $\Phi = \Phi_{\mathrm{max/hot}}$. If it is neither the cool ions or the hot ions that are imposing limits on the amplitudes of the solitons through the number density, the other alternative is that double layers, for which a positive or negative double root of $V(\Phi)$ coincides with where a local maximum of $V(\Phi)$ occurs, can also arise as upper limits on the admissible Mach number ranges for solitons. It has been found in Ref. 28 that when there is a switch in polarity of solitons (electron-acoustic in Ref. 28) from negative to positive, the negative polarity solitons and positive polarity solitons which occur in the regions which are adjacent to the cross over point are limited, respectively, by negative and positive double layers. Interestingly, the story on why the existence regions of solitons terminate does not end even here because for some models, solitons are even found beyond the M values for which double layers are supported,[19] i.e., supersolitons, and are ultimately limited by the constraint imposed by appropriate number density. Having provided in this discussion quick insights into how upper Mach number limits arise for solitons, we will now through numerical investigation identify the different reasons as to why upper Mach number limits occur for slow and fast ion-acoustic solitons, hence enabling us to present the existence regions of slow and fast ion-acoustic solitons in terms of the admissible Mach number ranges after considering where the lower and upper Mach number limits occur. In Sec. IV, we have presented the admissible slow and fast ion-acoustic soliton Mach number ranges as a function of the relative concentration and temperature of the cool ions with respect to the hot ions. These figures depicting the existence regions provide very useful insights into how the characteristics of slow and fast ion-acoustic solitons differ. The results depicted in Figures 2–6 and 8 and upper set of curves in Figures 7(a) and 7(b) in Sec. IV is based on the model which includes inertia and pressure for all species.

III. MODEL WITH BOLTZMANN HOT ELECTRONS

In this section, we present the model where the inertia (and pressure) of the cool and hot ions and the cool electrons is retained but the hot electrons are assumed to be isothermal and have a Boltzmann distribution. Equations (1)–(3) are all still valid for the cool ions, hot ions, and the cool electrons; however, the hot electron number density is now given by

$$n_{he} = n_{he}^0 \exp\left(\Phi/T_{he}\right) \tag{16}$$

and the corresponding Sagdeev potential contribution can be written as

$$V(\Phi, M)_{he} = n_{he}^0 T_{he}[1 - \exp\left(\Phi/T_{he}\right)]. \tag{17}$$

To obtain Sagdeev potential for this model, the inertial hot electron contribution in Eq. (11) needs to be replaced by the expression in Eq. (17). All other terms remain the same as in (11). Figure 9 and the lower set of curves in Figures 7(a) and 7(b) in Sec. IV are based on the model with Boltzmann hot electrons.

IV. NUMERICAL RESULTS AND DISCUSSION

As a precursor to investigating solitons, we have to first identify which linear wave modes are supported in our four-component model which is composed of cool ions, hot ions, and cool and hot electrons. The phase speeds of the linear wave modes which coincide with the roots corresponding to the vanishing of $C_2(M)$ (14), thus establish the existence of minimum values of the Mach number for which solitons associated with the different linear wave modes can exist.

We recall that C_2 is the second derivative of the Sagdeev potential which is evaluated at $\Phi = 0$. Numerically, we found three distinct roots of $C_2 = 0$ for a two-ion[12] model as opposed to only two in the one-ion[11] model. We associate the lowest and intermediate roots with the slow and fast ion-acoustic modes, whereas the largest root is the electron-acoustic mode. The roots obtained from $C_2 = 0$ are shown as a function of the cool ion number density in Figure 1. From Figure 1(a), it is clear that slow ion-acoustic modes have critical Mach numbers $M_{crit} = 0.17$ to 1.73 so that their phase speeds lie between the cool and hot ions thermal speeds. The critical Mach numbers for the fast ion-acoustic are between 2.3 and 4.5 (Figure 1(a)), therefore their phase speeds will lie between the hot ion and cool electron thermal velocities. From Figure 1(b), it is seen that critical Mach numbers for electron-acoustic modes are in the range of 90–180, and correspondingly their phase speeds lie between the cool electron and hot electron thermal speeds. The phase speed of electron-acoustic modes is not sensitive to variations in cool ion number density. This is expected as this is a high-frequency mode where ion dynamics does not play an important role; it merely provides a neutralizing background. The characteristics of the slow and fast ion-acoustic waves from the phase speeds which we obtained numerically are in agreement with the behaviour of the modes inferred from analytical expressions for the frequencies of the slow and fast ion-acoustic modes (Refs. 32–34). We have verified that when the temperatures of the two ions are the same, the slow ion-acoustic mode disappears and we get only the fast ion-acoustic mode.

Large amplitude solitons of the ion-acoustic type for which existence regions are shown as a function of the cool ion number density, viz., n_{ci0}/n_0 in Figure 2, are referred to as "slow ion-acoustic solitons." The solid (–) curve in Figure 2 shows how the critical value of the Mach number for slow ion-acoustic solitons varies with cool ion number density. For a fixed value of n_{ci0}/n_0, slow ion-acoustic solitons will occur for Mach number values which exceed the critical value which lies on the solid (–) curve but will terminate once the upper limit on M has been reached which lies on either one of the curves denoted by $(\cdots)$, (- -), $(-\cdot-)$, or $(-\cdots\cdot-)$, depending on the particular choice of n_{ci0}/n_0. As a quick aside, it is important to take note that slow ion-acoustic solitons are not possible if there are no cool ions in the model, because the linear mode is non-existent.[34] This point is very clearly realised in Figure 2 and later in Figure 9 where it is seen that the lower (–) and upper Mach number limits $(\cdots)$ tend to converge as $n_{ci0}/n_0 \to 0$. A comparison of Figure 2 with Figure 7 which depicts existence regions of fast ion-acoustic solitons shows that fast ion-acoustic solitons depend on the presence of hot ions in the model and obviously do not vanish when $n_{ci0}/n_0 \to 0$.

We consider first the soliton region which lies above the curve denoted by (–) representing the lower M limits but below the upper M limits ($\cdots$, region I) in Figure 2, by fixing the value of the cool ion density at a low value $n_{ci0}/n_0 = 0.1$. The critical value of the Mach number $M = 0.532354$ which coincides with the point (corresponding to $n_{ci}/n_0 = 0.1$) on (–) in Figure 2 represents a lower limiting value for solitons. It can be clearly seen that the value $M = 0.532354$ (–) for which $V(\Phi)$ does not have a positive root provides a lower M limit for the solitons shown in Figure 3. For a possible soliton solution, the Mach number values have to exceed the critical value which is clearly shown in Figure 3 for $M = 0.56$ $(\cdots)$, $M = 0.6$ $(--)$, and $M = 0.62$ $(-\cdot-)$. These are seen to become stronger

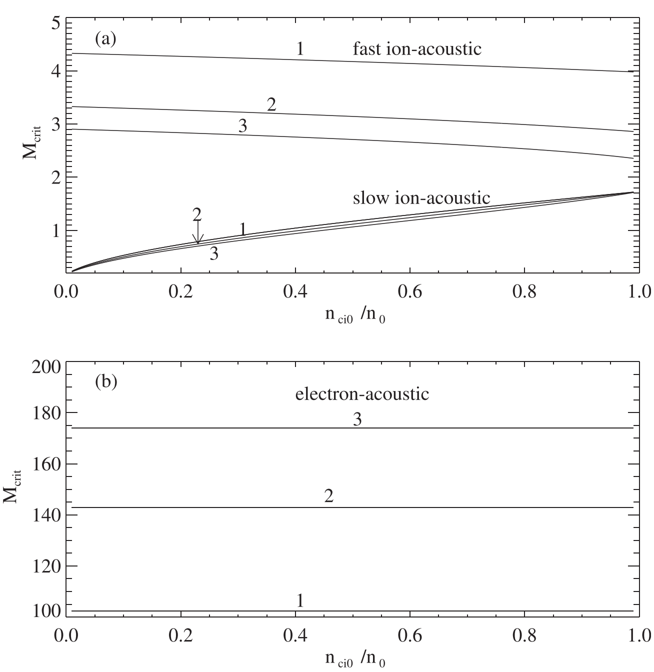

FIG. 1. Phase speeds of the linear (a) slow and fast ion-acoustic and (b) electron-acoustic wave modes as a function of the cool ion number density for the fixed parameters $\mu_e = 1/1836$, $T_{ce}/T_{hi} = 1$, $T_{he}/T_{hi} = 10$, and $T_{ci}/T_{hi} = 0.01$. The curves correspond to $n_{ce0}/n_0 = 0.1$ (curve labelled 1), 0.3 (curve labelled 2), and 0.5 (curve labelled 3).

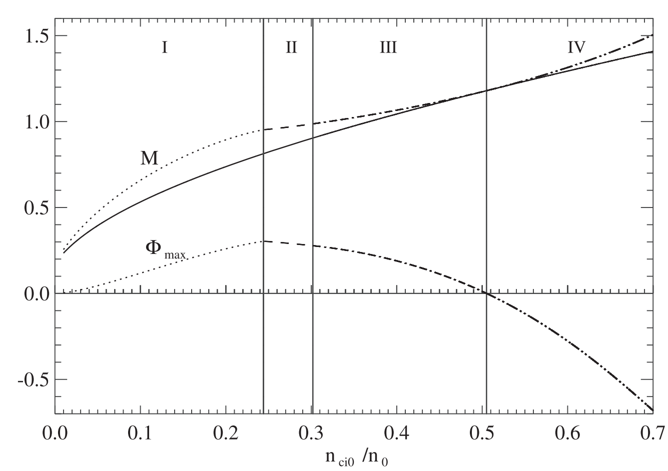

FIG. 2. Existence regions of slow ion-acoustic solitons shown as a function of the normalized cool ion number density for the fixed parameters: $\mu_e = 1/1836$, $T_{ce}/T_{hi} = 1$, $T_{he}/T_{hi} = 10$, $T_{ci}/T_{hi} = 0.01$, and $n_{ce0}/n_0 = 0.1$. The lower (–) and upper Mach number limits beyond which the cool (cf. Eq. (6)) ($\cdots$, region I) and hot (cf. Eq. (7)) (- -, region II) ion number densities become complex valued, and for which positive and $(-\cdot-$, region III) and negative $(-\cdots\cdot-$, region IV) double layers are supported and the corresponding maximum potentials (Φ_{max}) ($\cdots$, region I) and (- -, region II) and positive $(-\cdot-$, region III) and negative $(-\cdots\cdot-$, region IV) double layer amplitudes are shown.

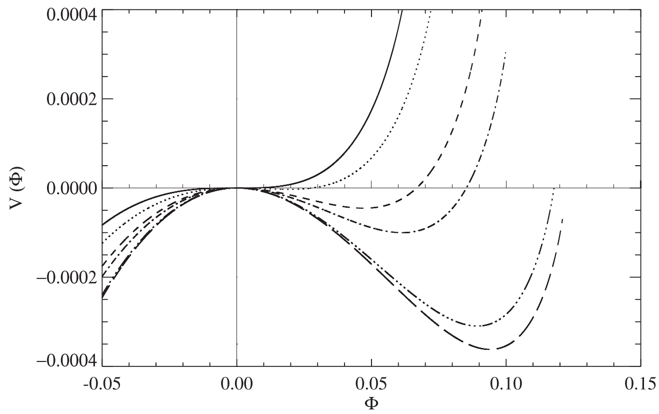

FIG. 3. The Sagdeev potential for $M = 0.532354$ (–), $M = 0.56$ $(\cdots)$, $M = 0.6$ (- -), $M = 0.62$ $(- \cdot -)$, $M = 0.6585157$ $(- \cdots -)$, and $M = 0.665$ (– –). The other fixed parameters are same as in Figure 2 except $n_{ci0}/n_0 = 0.1$.

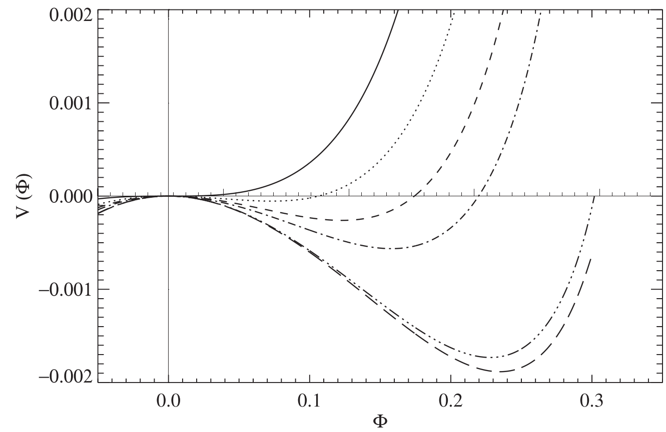

FIG. 4. The Sagdeev potential for $M = 0.82296736$ (–), $M = 0.87$ $(\cdots)$, $M = 0.9$ (- -), $M = 0.92$ $(- \cdot -)$, $M = 0.955187$ $(- \cdots -)$, and $M = 0.958$ (– –). The other fixed parameters are same as in Figure 3 except $n_{ci0}/n_0 = 0.25$.

(amplitudes increase) as inferred from the positive roots of $V(\Phi)$ which become more positive for increasing values of the Mach number. This increase in soliton amplitude with increasing Mach number will not continue indefinitely. Eventually the upper limit on the Mach number for slow ion-acoustic solitons will be realized in Figure 3 when the limiting curve $(- \cdots -)$ for the Sagdeev potential which corresponds to $M = 0.6585157$ is obtained. If we examine the expression for the cool ion number density (6), it is apparent from the second term that for positive values of the potential, Φ should not exceed $\Phi_{\max/\mathrm{cool}} = (M - \sqrt{3T_{ci}})^2/2$ for the cool ion number density to remain real valued. The root of $V(\Phi) = 0$ corresponding to the upper limiting value of M coincides precisely with this limit on Φ, viz., $\Phi_{\max/\mathrm{cool}} = (M - \sqrt{3T_{ci}})^2/2$ such that (6) is still real valued. Reverting to our Figure 2 which depicts the slow ion-acoustic soliton existence regions, solitons will occur for M values which lie above (–) but are below $(\cdots$, region I), where the latter upper M limits are imposed by the constraint that the number density of the cool ions has to remain real valued. Slow ion-acoustic solitons cease to exist for a value for M which either lies on $(\cdots)$ in Figure 2 or above it such as for $M = 0.665$ (– –) in Figure 3 in which case $V(\Phi)$ no longer has a positive root. The upper limiting values of M which lie on $(\cdots)$ in Figure 2 were obtained numerically by solving $V(\Phi = \Phi_{\max/\mathrm{cool}}) = 0$. The limit for positive values of the potential, viz., $\Phi_{\max/\mathrm{cool}} = 0.1177632$ beyond which the cool ion number density (6) is no longer real valued, coincides with the positive root of $V(\Phi)$ for $M = 0.6585157$ $(- \cdots -)$ in Figure 3. It is clearly seen in Figure 3 that slow ion-acoustic solitons cease to exist and $V(\Phi)$ (– –) will no longer have a positive root once M exceeds the upper limit $M = 0.6585157$. Values of $\Phi_{\min/\mathrm{cool}}$ $(\cdots$, region I) are shown in the lower set of curves in Figure 2 for the entire range of cool ion concentrations n_{ci0}/n_0 where the cool ions are responsible for the upper limit.

We move now to the second region in parameter space where the upper Mach number limits on (- -, region II) in Figure 2 are imposed by the constraint that the number density of the hot ions (7) must remain real valued. Plots of Sagdeev potentials in Figure 4 for the higher value $n_{ci0}/n_0 = 0.25$ rather than $n_{ci0}/n_0 = 0.1$ (in Figure 3) look quite similar to the Sagdeev potentials in Figure 3 except that the lower and upper Mach number limits are different from that in Figure 3. The value $M = 0.82296736$ for the critical Mach number corresponding to $n_{ci0}/n_0 = 0.25$ which gives rise to the lower limiting plot (–) of the Sagdeev potential in Figure 4 lies on (–) in Figure 2. On the other hand, the value $M = 0.955187$ (for $n_{ci0}/n_0 = 0.25$) which gives rise to the upper limiting plot of the Sagdeev potential $(- \cdots -)$ in Figure 4 lies on the upper limiting curve (- -, region II) for M in Figure 2. The upper limit on M coincides with the maximum permitted value of Φ, viz., $\Phi_{\max/\mathrm{hot}} = (M - \sqrt{3})^2/2$ beyond which the number density of the hot ions (7) is no longer real valued. The variation of $\Phi_{\max/\mathrm{hot}}$ with n_{ci0}/n_0 is shown as (- -, region II) in the lower set of curves in Figure 2. The upper limits on M (- -, region II) in Figure 2 got from $V(\Phi = \Phi_{\max/\mathrm{hot}}) = 0$ (as opposed to $V(\Phi = \Phi_{\max/\mathrm{cool}}) = 0$) were in agreement with the M values which result in the upper limiting curves of the Sagdeev potentials. This led us in the right direction that the hot ion number density is responsible for limiting the occurrence of slow ion-acoustic solitons in this region II of parameter space. Once the upper limit $M = 0.955187$ corresponding to $n_{ci0}/n_0 = 0.25$ has been exceeded, slow ion-acoustic solitons are no longer possible such as for the value $M = 0.958$ (– –) in Figure 4 because the Sagdeev potential becomes complex valued long before a positive root of $V(\Phi)$ can be attained, similar to what we discussed earlier when the cool ion number density was responsible for the upper limit.

Reverting to the existence regions in Figure 2, the upper M limits for slow ion-acoustic solitons which lie on $(- \cdot -$, region III) in the range $0.3019 \le n_{ci0}/n_0 < 0.505$ are not Mach number values beyond which either the number density of the cool ions (6) or the hot ions (7) becomes complex valued, but these are now Mach number values for which positive potential double layers occur. A double layer has an asymmetric potential profile for which the behaviour of the Sagdeev potential is such that $\mathrm{d}V(\Phi)/\mathrm{d}\Phi = 0$ coincides with the position of a double (non-trivial) root of $V(\Phi)$. It can clearly be seen in Figure 5 that the double layer which was obtained for the value $M = 1.0663942$ $(- \cdots -)$ provides the upper limit on the M range supporting slow ion-acoustic solitons found for $n_{ci0}/n_0 = 0.4$. The amplitude of the double layer, viz., $\Phi = 0.1895192$ coincides with the point corresponding to $n_{ci0}/n_0 = 0.4$ on ($- \cdot -$, region III) in the lower set of curves in Figure 2. It is clearly seen in Figure 5 that beyond the Mach number value for which a positive double layer is supported, slow ion-acoustic solitons are no longer possible.

Finally, for the largest cool ion concentrations spanning $0.5053 \leq n_{ci0}/n_0 \leq 0.7$, slow ion-acoustic solitons are found to have negative potentials as opposed to positive potentials found for solitons in the three regions discussed earlier. Fixing the value for $n_{ci0}/n_0 = 0.65$ (Figure 6), the existence of these negative polarity solitons starts at a value for the Mach number $M = 1.351656$, which lies on (–) in Figure 2, they become stronger (amplitudes become increasingly more negative) with increasing values of the Mach number and their existence terminates on the upper limiting curve ($- \cdots -$, region IV) at the value $M = 1.4021812$ which is a negative potential double layer. Our results are quite similar to the findings for two negatively charged inertial species (electrons)[9,10] in that, if one is warmer than the other, a switch in polarity can be incurred by varying the concentration of the cooler species. In fact, the pattern for our existence regions depicted in Figure 2, is very similar to those shown for electron-acoustic solitons in Refs. 9 and 28 in that soliton regions limited by the cool and hot number densities (of the electrons) is followed by a double layer region (soliton structures limited by double layers have the same polarity as the two inertial electron species) and this is followed by another double layer region (nonlinear structures now have opposite polarity to that of the two inertial electron species). Reverting to our discussion of Figure 2 and examining the lower set of curves, the cool ion concentration at which there is a cross-over from positive ($- \cdot -$, region III) polarity solitons to negative ($- \cdots -$, region IV) polarity solitons, which is observed to occur at the value $n_{ci0}/n_0 = 0.505$ in Figure 2 depends on the particular fixed concentration of the cool electron density. We have defined the critical value of the cool ion number density as being the cool ion

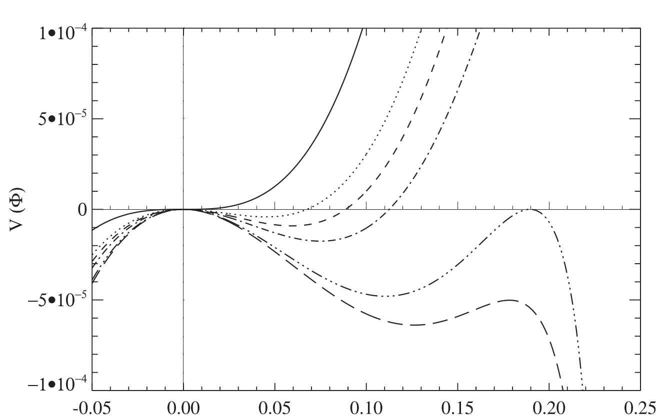

FIG. 5. The Sagdeev potential for $M = 1.0439219$ (–), $M = 1.055$ ($\cdots$), $M = 1.058$ (- -), $M = 1.061$ ($- \cdot -$), $M = 1.0663942$ ($- \cdots -$), and $M = 1.068$ (– –). The other fixed parameters are same as in Figure 2 except $n_{ci0}/n_0 = 0.4$.

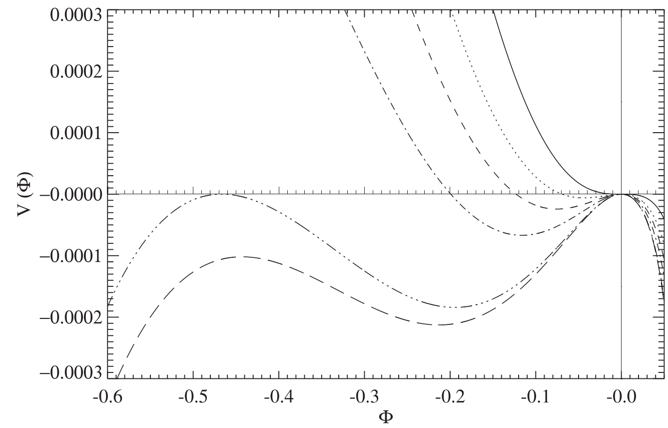

FIG. 6. The Sagdeev potential for $M = 1.351656$ (–), $M = 1.37$ ($\cdots$), $M = 1.38$ (- -), $M = 1.39$ ($- \cdot -$), $M = 1.4021812$ ($- \cdots -$), and $M = 1.404$ (– –). The other fixed parameters are same as in Figure 2 except $n_{ci0}/n_0 = 0.65$.

concentration which corresponds to the vanishing of the third derivative of the Sagdeev potential (at $M_{\rm crit}$) evaluated at $\Phi = 0$. Positive $(d^3V(\Phi/d\Phi^3)_{\Phi=0} > 0)$ and negative $(d^3V(\Phi/d\Phi^3)_{\Phi=0} < 0)$ potential KdV solitons having solutions described by (13) occur on either of the critical value for the cool ion concentration. The critical value of the cool ion concentration at which a switch from positive to negative potential slow ion-acoustic solitons will occur increases from $n_{ci0}/n_0 = 0.468$ at $n_{ce0}/n_0 = 0$ to $n_{ci0}/n_0 = 0.505$ at $n_{ce0}/n_0 = 0.1$. For a higher value of the cool electron concentration, viz., $n_{ce0}/n_0 = 0.2$ there is a switch from positive polarity solitons to negative polarity solitons at $n_{ci0}/n_0 = 0.541$.

Having discussed in detail the existence regions of slow ion-acoustic waves, we now shift our focus to fast ion-acoustic solitons. These will simply be referred to as ion-acoustic solitons for the sake of brevity. The picture of the ion-acoustic soliton existence regions shown in Figure 7 for

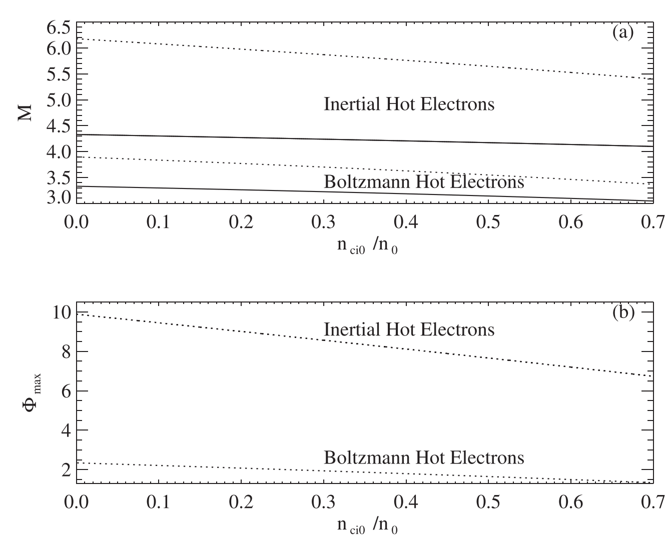

FIG. 7. Existence regions of fast ion-acoustic solitons shown as a function of the normalized cool ion number density for both inertial and Boltzmann hot electrons. The fixed parameters are same as in Figure 2. The lower (–) and upper Mach number limits beyond which hot ion number density (cf. Eq. (7)) ($\cdots$) becomes complex valued are shown in (a). The corresponding maximum values of the potentials ($\cdots$) are shown in (b).

inertial, adiabatic hot electrons is very much simpler than the picture shown for slow ion-acoustic solitons in Figure 2 because there is only one reason as to why upper M limits occur for ion-acoustic solitons. We conclude that only positive potential ion-acoustic solitons are possible for our model in view of the fact that the third derivative (15) of the Sagdeev potential (11) at $\Phi = 0$ remains positive irrespective of how the parameters are varied. It turns out that it is the hot ions and not the cool ions which are responsible for the upper M limits for fast ion-acoustic solitons. The hot ion number density establishes the existence of a limiting value of the potential, viz., $\Phi_{\rm max/hot} = (M - \sqrt{3})^2/2$, such that, if $\Phi > \Phi_{\rm max/hot}$, then (7) becomes complex valued if the Mach number exceeds the upper M limit on $(\cdots)$ in Figure 7(a). The variation of $\Phi_{\rm max/hot}$ with the cool ion number density for adiabatic hot electrons is depicted in Figure 7(b). An interesting comparison of our results for fast ion-acoustic solitons with the results of Mishra *et al.*[33] is that both negative and positive polarities were found for fast ion-acoustic solitons in a negative-ion plasma when the cool and hot electron species are Boltzmann distributed. The results in the studies by Baboolal *et al.*[17,18] for a two-ion model discusses only ion-acoustic solitons and/or double layers associated with the fast ion-acoustic mode. The discussion of the results for fast ion-acoustic solitons depicted in Figure 7 for Boltzmann hot electrons will be deferred until later.

Up to this point, our focus was on examining the slow ion-acoustic soliton existence regions as a function of the cool ion number density by keeping the cool to hot ion temperature ratio fixed at $T_{ci}/T_{hi} = 0.01$. By fixing the cool ion number density at the value $n_{ci0}/n_0 = 0.3$, and increasing T_{ci}/T_{hi}, our Figure 8 reveals that there is a switch from a region where the positive polarity soliton amplitude restrictions are due to the occurrence of positive potential double layers, followed by the region where the soliton amplitude restrictions are imposed by the number density constraints imposed by the hot ions and then the cool ions. These three mentioned regions are, respectively, bounded by the upper Mach number limiting curves denoted by $(\cdots$, region I), (- -, region II), and $(- \cdot -$, region III) in Figure 8. Values for Φ_{max} are found to decrease with the increase of T_{ci}/T_{hi} as seen in the lower set of curves in Figure 8. We did consider T_{ci}/T_{hi} values up to $T_{ci}/T_{hi} = 0.9$ and found that the cool ion number density limit extends all the way up to $T_{ci}/T_{hi} = 0.9$. We have not shown the entire range in Figure 8 because it would not have been possible to show the positive double layer and the hot ion number density limited regions which are much narrower.

Finally, we consider the case of treating the hot electrons as inertialess in our model as opposed to including the inertia of the hot electrons. At first glance, it will seem that the slow ion-acoustic soliton existence regions in Figure 9 are identical to those shown in Figure 2, however, a closer comparison of the figures particularly at the larger values of n_{ci0}/n_0 where negative potential solitons occur which are limited by negative double layers, it will become evident that the admissible soliton Mach number ranges are narrower when the hot electrons are treated as Boltzmann as opposed to retaining inertia for the hot electrons. Because the upper (negative double layer) limits are so close to the $M_{\rm crit}$ values, this justifies why the negative double layer amplitudes $(- \cdots -$, region IV) in the region of high cool ion concentrations in Figure 9 are also smaller than those shown for inertial hot electrons in Figure 2 if one compares the negative double layers for the same cool ion concentrations. Another noticeable difference if one compares Figure 9 with Figure 2 is that the cross-over from positive polarity to negative

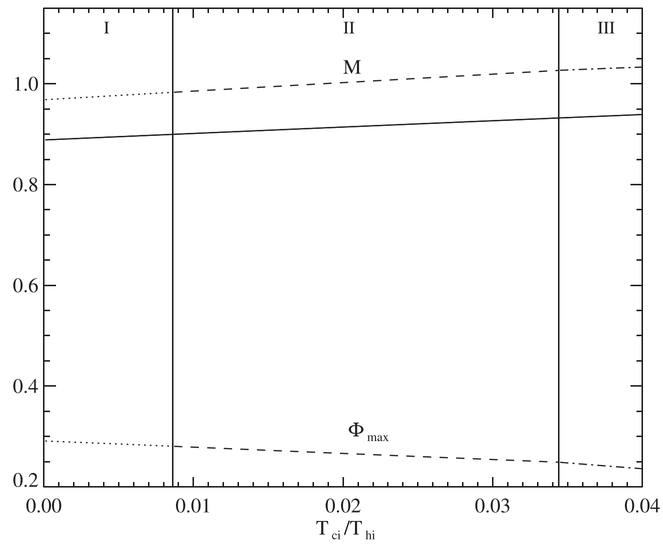

FIG. 8. Existence regions of slow ion-acoustic solitons shown as a function of the cool to hot ion temperature ratio. The fixed parameters are same as in Figure 3 except $n_{ci0}/n_0 = 0.3$. The lower Mach number limit (–), and upper Mach number limits for which positive double layers $(\cdots$, region I) occur, and upper Mach number values beyond which the hot (cf. Eq. (7)) (- -, region II) and cool (cf. Eq. (6)) $(- \cdot -$, region III) ion number densities become complex valued and the corresponding positive double layer amplitudes $(\cdots$, region I), and maximum potentials (- -, region II) and $(- \cdot -$, region III) are shown.

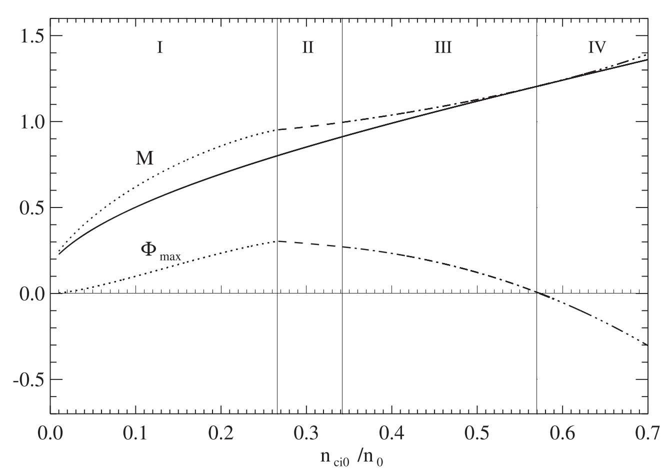

FIG. 9. Existence regions of slow ion-acoustic solitons shown as a function of the normalized cool ion number density for Boltzmann hot electrons. The fixed parameters are same as in Figure 2. The lower (–) and upper Mach number limits beyond which the cool ion (cf. Eq. (6)) $(\cdots$, region I) and hot ion (cf. Eq. (7)) (- -, region II) number densities become complex valued, and Mach number values for which positive $(- \cdot -$, region III) and negative $(- \cdots -$, region IV) double layers occur and the corresponding maximum values of the potentials $(\cdots$, region I) and (- -, region III) and positive $(- \cdot -$, region III) and negative $(- \cdots -$, region IV) double layer amplitudes are shown.

polarity structures occurs at a larger value for the cool ion density, viz., $n_{ci0}/n_0 = 0.57$ than the value $n_{ci0}/n_0 = 0.505$ found for the case of inertial hot electrons for the same value $n_{ce0}/n_0 = 0.1$ for the cool electron concentration.

A comparison of the fast ion-acoustic soliton existence regions shown in Figure 7 reveals that there are no qualitative changes for Boltzmann and inertial hot electrons. Fast ion-acoustic solitons are again found to be limited only by the constraint that the number density of the hot ions (7) has to remain real valued, which is consistent with what we found when the hot electrons are inertial in the model (Figure 7). Figure 7 clearly shows that the admissible ion-acoustic soliton Mach number ranges are much wider, consequently, the maximum permitted values of the potential beyond which the hot ion number density (7) is no longer real valued, viz., $\Phi_{max/hot}$ are much higher when hot electron inertia is retained as opposed to neglecting it in the theoretical model. If one considers the results for electron-acoustic solitons for previously studied models with three[10] or two[28] electron components, then the effect of including the inertia for two electron components in the model was found to allow for positive polarity electron-acoustic structures to be possible. In a similar vein, we expect that negative polarity ion-acoustic solitons will not be possible if the hot ions are treated as Boltzmann in the two-ion model, although we did not investigate this in this study. This will only allow for the possibility of a single ion-acoustic mode, the hot ion number density restriction will be removed, the possibility of a polarity switch for ion-acoustic solitons will be eliminated, and consequently, the occurrence of double layers will not be supported. If there are two inertial species in the model (electrons[9,28] or ions[34]), earlier studies[9,28,34] and this study have shown that double layers (DLS) are possible in adjacent regions of parameter space which lie on either side of where the polarity switch in the supported soliton structures occurs. Certainly, if the inertia of the cool and hot electrons is neglected in the theoretical model, then a polarity switch is very likely for ion-acoustic solitons[17] although we are not including this possibility for the purpose of this discussion. We should, however, point out that caution should be exercised in considering the hot ions to be Boltzmann distributed on ion time scales, since this approximation will only be a good one if the cool ions are very much more massive than the hot ions in the model, consequently, this may be better suited to dusty plasma models.

V. CONCLUSIONS

The existence of large amplitude slow and fast ion-acoustic solitons has been investigated for a model with cool ions, hot ions, cool electrons, and hot electrons using the Sagdeev pseudo-potential formalism. This model is applicable to the multispecies plasma found in the plasma sheet boundary layer[29] of the Earth's magnetosphere. In the study by Lakhina *et al.*,[12] it was established that two ion-acoustic modes having different phase speeds are supported when the model with only one[11] species of ion was extended to include two species of inertial ions, viz., cool and hot. The purpose of this study was to broaden the scope of the findings in Ref. 12 by focusing on why upper Mach number limitations arise for slow and fast ion-acoustic solitons. Furthermore, the parameters which were chosen in this study do not assume the hot ions to be much hotter than the hot electrons which will result in reduced Landau damping of the linear wave modes, and this will consequently favour the plasma evolving to a nonlinear state quickly enough, such that solitons and double layers are supported. Treating the cool and hot electrons as adiabatic fluids in our study rather than as isothermal and inertialess species (Boltzmann distributed) is justified in situations where the hot ions are either much hotter[29] or have the same temperature as the cool electrons such that there is no wide separation between the thermal speeds of both species. We find that initially, retaining inertia and pressure for all species, in the order of increasing concentrations of the cool ions, viz., n_{ci0}/n_0, slow ion-acoustic solitons having positive polarity are limited by the cool and then the hot ion number density becoming complex valued, followed by the region where the positive polarity slow ion-acoustic solitons are limited by positive double layers. The largest cool ion concentrations induce a polarity switch in slow ion-acoustic solitons such that negative potential structures become possible and these are limited by negative double layers. The higher phase speed (fast) ion-acoustic solitons in the study were found to have only positive polarity and these are limited only by the constraint that the number density of the hot ions should remain real valued. Both polarities were found for fast ion-acoustic solitons in a negative-ion plasma with Boltzmann cool and hot electrons.[33] The admissible negative potential slow ion-acoustic Mach number ranges at the higher cool ion concentrations were found to be narrower, consequently the negative double layers limiting the existence regions of negative slow ion-acoustic solitons have reduced amplitudes for the model with Boltzmann hot electrons as compared to the nonlinear structures obtained if the hot electrons are treated as inertial. The other effect of neglecting as opposed to retaining inertial effects of the hot electrons is that the cross over from positive to negative potential slow ion-acoustic solitons occurs at a higher value of the cool ion number density than for the model with inertial hot electrons. The effect of neglecting as opposed to retaining inertial effects of the hot electrons on the existence regions of fast ion-acoustic solitons is that not only is there a shift in the admissible Mach number ranges to lower values but also these Mach number ranges which support fast ion-acoustic solitons are narrower for hot electrons which are Boltzmann distributed than for inertial hot electrons.

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