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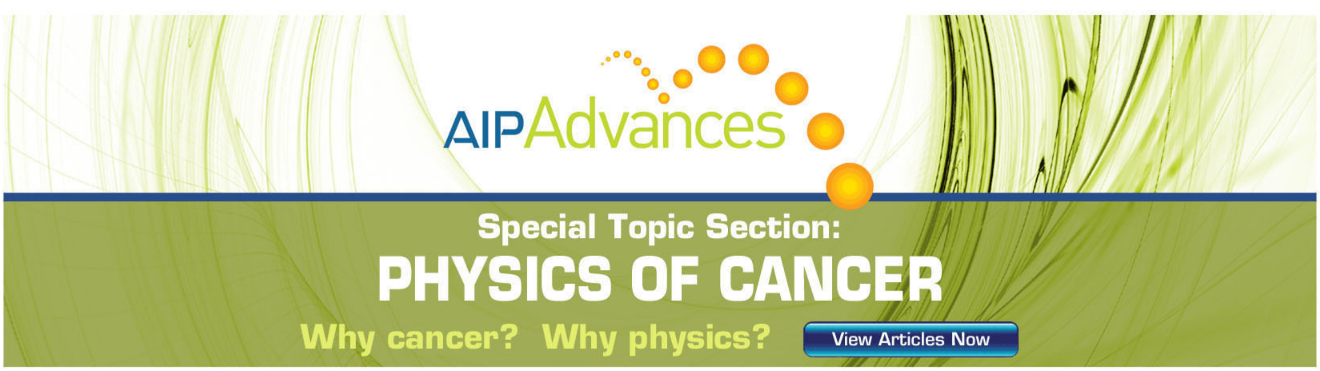

Existence domains of arbitrary amplitude nonlinear structures in two-electron temperature space plasmas. I. Low-frequency ion-acoustic solitons

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Using the Sagdeev pseudopotential technique, the existence of large amplitude ion-acoustic solitons is investigated for a plasma composed of ions, and hot and cool electrons. Not only are all species treated as adiabatic fluids but the model for which inertial effects of the hot electrons is neglected whilst retaining inertia and pressure for the ions and cool electrons has also been considered. The focus of this investigation has been on identifying the admissible Mach number ranges for large amplitude nonlinear ion-acoustic soliton structures. The lower Mach number limit yields a minimum velocity for the existence of ion-acoustic solitons. The upper Mach number limit for positive potential solitons is found to coincide with the limiting value of the potential (positive) beyond which the ion number density ceases to be real valued, and ion-acoustic solitons can no longer exist. Small amplitude solitons having negative potentials are found to be supported when the temperature of the cool electrons is negligible. © *2012 American Institute of Physics*.
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I. INTRODUCTION

Nonlinear solitary wave structures having the form of travelling bipolar pulses in the magnetic field-aligned component of the electric field have been observed in different regions of the terrestial magnetosphere such as the mid-altitude auroral zone,[1] polar magnetosphere,[2] bow shock,[3] plasma sheet boundary layer,[4] and more recently in the dayside magnetosheath.[5] The short durations of the periods of the observed pulses indicate that these structures are related to electron dynamics. The relevance of the observed nonlinear waveforms to the high frequency portion of the spectrum of broadband electrostatic noise (BEN) has been pointed out.[1,2,4,5] On the other hand, nonlinear structures, such as solitary waves and weak double layers having negative potentials, observed in the auroral acceleration region by S3-3 (Ref. 6) and Viking,[7,8] were found to be fluctuations associated with ion dynamics. In view of these observations, there have been several attempts to reconcile the occurrence of these nonlinear wave structures with suitable theoretical models.

There are numerous reports on nonlinear studies on ion-acoustic solitons and double layers. Large and small amplitude ion-acoustic double layers have been investigated by Baboolal *et al.*[9] for a plasma composed of hot and cool Boltzmann electrons, and two species of warm fluid ions (both positive). Existence domains of arbitrary amplitude ion-acoustic solitons and double layers have been presented in Baboolal *et al.*[10] not only for the model of Baboolal *et al.*[9] where both ions are positive but also for a negative-ion plasma where one of the ion species is negatively charged. Both negative and positive potential soliton solutions were found in Ref. 10. It was found that if both ion species have positive polarity, negative double layers were found to limit the existence domain of negative potential ion-acoustic solitons,[10] however this was found not to be the case if one of the ion species is negatively charged. Both large amplitude and small amplitude ion-acoustic double layers have been investigated by Bharuthram and Shukla[11] for a model composed of cool ions, and hot and cool electrons where both electron species are Boltzmann distributed. Although the existence of large amplitude ion-acoustic solitons was not investigated in Ref. 11, we know from Ref. 10 that the large amplitude negative potential ion-acoustic double layers found in Ref. 11 mark the end of the Mach number regimes supporting large amplitude negative potential ion-acoustic solitons.

Large amplitude ion-acoustic solitons were investigated for a plasma composed of cool ions, hot ions, and electrons by Hellberg and Verheest[12] using the fluid-dynamic paradigm[13–15] where thermal effects of the cool ion species and inertial effects of the electrons and the hot ions were neglected. Only positive potential ion-acoustic solitons were found to be supported for which the admissible Mach number ranges are presented. For small concentrations of the cool ions, the upper Mach number limit was imposed by the cool ions (cool ion number density becomes infinite), whereas for higher cool ion concentrations, the hot ions (hot ion number density goes to zero) were found to be responsible for the upper limit for the value $\gamma = 2$ for the polytropic indices of the electrons and the hot ions, provided that the ratio of the temperatures of the hot ions to the electrons is not very large. For the value $\gamma = 1$ (Boltzmann limit) for the polytropic indices of the electrons and the hot ions, only the cool ions were found to be responsible for the upper Mach number limit.

In a more recent investigation by Verheest and Hellberg,[16] the Sagdeev potential formalism was adopted to investigate the existence of large amplitude ion-acoustic solitons in a two-ion plasma where pressure and inertia is retained for both ions species, where the cool (smaller thermal speed) positive ions are supersonic (speed of the nonlinear structure exceeds the thermal speed of the cool ions), whereas the negative hot (larger thermal speed) ions are subsonic (the thermal speed of the hot ions exceeds the speed of the nonlinear structure). The focus of the study was on obtaining the permitted velocity ranges for which ion-acoustic solitons occur. Consistent with the polarity of the supersonic ion species, only positive potential solitons were found, where the restriction on the maximum attainable potential (positive) of the nonlinear structures was found to be imposed by the existence of a limiting value of the positive potential beyond which the number density of the cool ions ceases to be real valued.

The existence of arbitrary amplitude ion-acoustic and electron-acoustic solitons has been investigated by Lakhina *et al.*[17] for a plasma composed of ions and cool and hot electrons, where all species are assumed to be adiabatic fluids. The critical values of the Mach number for ion-acoustic and electron-acoustic solitons were obtained. Only positive potential ion-acoustic solitons were found, however electron-acoustic solitons having either negative or positive polarity were found to be possible. Although, the existence of an upper limit on the permissible Mach numbers for large amplitude ion-acoustic and electron-acoustic solitons was mentioned in Ref. 17, the upper Mach number limits where the existence domains of ion-acoustic and electron-acoustic solitons terminate were not calculated. Furthermore, the physical mechanism for the existence of upper limiting values of the Mach number for ion-acoustic and electron-acoustic solitons was not discussed in Ref. 17.

Here, we focus on understanding why upper Mach number limits exist for solitons and we explicitly determine these upper limits for large amplitude ion-acoustic and electron-acoustic solitons but for much broader regions in parameter space than those investigated in Ref. 17. Taking both the lower and upper Mach number limits into consideration, we present the Mach number ranges, which support the existence of ion-acoustic and electron-acoustic solitons. Our findings based on the model of Lakhina *et al.*[17] which assumes finite inertia for the hot electrons are compared with the results we have obtained using the model of Mace *et al.*[18] We recall that inertial effects of the hot electrons are not considered in the model of Ref. 18, wherein the existence of only electron-acoustic solitons is discussed. So as not to overload this paper, we have divided our findings into two parts, *viz*, Part I and Part II. In the first part of our study which is titled "Existence domains of arbitrary amplitude nonlinear structures in two-electron temperature space plasmas: I. Low-frequency ion-acoustic solitons" (hereafter cited as I), we present our results pertaining to large amplitude ion-acoustic solitons. In the second part of our study entitled "Existence domains of arbitrary amplitude nonlinear structures in two-electron temperature space plasmas: II. High-frequency electron-acoustic solitons" (hereafter cited as II), we discuss our findings pertaining to large amplitude electron-acoustic solitons.

The outline of the paper is as follows. In Sec. II, we present details of the theory for the three-component model composed of ions, cool electrons, and hot electrons, for which inertia and pressure are included for all species.[17] The model which does not take into consideration inertial effects of the hot electrons[18] is discussed in Sec. III. Existence domains of large amplitude ion-acoustic solitons are presented and discussed in Sec. IV. Finally, a summary of our findings and conclusions appears in Sec. V.

II. MODEL AND GOVERNING EQUATIONS TAKING INTO CONSIDERATION INERTIAL EFFECTS OF THE HOT ELECTRONS

We consider an unmagnetized plasma composed of ions, cool electrons, and hot electrons. Including inertia and pressure for all three plasma species, the continuity, momentum, and pressure equations for all three species are given in Ref. 17 as

$$\frac{\partial n_j}{\partial t} + \frac{\partial (n_j v_j)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial v_j}{\partial t} + v_j \frac{\partial v_j}{\partial x} = -\frac{Z_j}{\mu_j}\frac{\partial \Phi}{\partial x} - \frac{1}{\mu_j n_j}\frac{\partial P_j}{\partial x}, \tag{2}$$

$$\frac{\partial P_j}{\partial t} + v_j \frac{\partial P_j}{\partial x} + 3P_j \frac{\partial v_j}{\partial x} = 0, \tag{3}$$

and

$$\frac{\partial^2 \Phi}{\partial x^2} = n_{ce} + n_{he} - n_i, \tag{4}$$

where n_j, v_j, T_j, $P_j = n_j T_j$ denotes, respectively, the normalized number density, fluid velocity, temperature, and pressure of species j where $j = i,\ ce$ and he, respectively, denotes the ions, cool electrons, and hot electrons. Furthermore, Φ is the normalized wave potential, $\mu_{ce} = \mu_{he} = \mu_e = m_e/m_i$, where m_j denotes the mass of species j, $\mu_i = 1, Z_{ce} = Z_{he} = -1$ for the cool (or hot) electrons, and $Z_i = 1$. All densities are normalized with respect to the total equilibrium electron (or ion) number density, *viz*, $n_{i0} = n_{ce0} + n_{he0}$, velocities are normalized with respect to the ion thermal speed $C_i = (T_i/m_i)^{1/2}$, time with respect to the inverse ion plasma frequency $\omega_{pi}^{-1} = (m_i/4\pi n_{i0} e^2)^{1/2}$, lengths with respect to the ion Debye length $\lambda_{di} = (T_i/4\pi n_{i0} e^2)^{1/2}$, potential with respect to T_i/e, and thermal pressures with respect to $n_{i0}T_i$. Assuming an adiabatic fluid response of all species, the same value for the polytropic index, *viz*, $\gamma = 3$, has been used for all species.

We transform the set of Eqs. (1)–(4) to a frame moving with the wave through the co-moving co-ordinate $\xi = x - Mt$, where $M\ (= V/C_i)$ denotes the speed of the nonlinear wave structures, normalized with respect to the ion thermal speed, simply known as the Mach number. Following the mathematical procedure in Mendoza-Briceño *et al.*,[19] we solve for the densities for the different species. We make use of the boundary conditions for localized nonlinear wave solutions given by

$$\Phi \to 0, \quad \frac{d\Phi}{d\xi} \to 0, \quad n_i \to 1, \quad n_{ce} \to n_{ce}^0, \quad n_{he} \to n_{he}^0,$$
$$P_i \to 1, \quad P_{ce} \to n_{ce}^0 T_{ce}, \quad P_{he} \to n_{he}^0 T_{he} \text{ as } |\xi| \to \pm\infty, \tag{5}$$

where $n_j^0 = n_{j0}/n_{i0}$ such that $n_{ce}^0 + n_{he}^0 = n_i^0 = 1$.

For the number densities of the ions, cool and hot electrons, we have obtained the expressions given by

$$n_i = \frac{1}{\sqrt{6}}\left\{(M^2+3-2\Phi) \pm \sqrt{(M^2+3-2\Phi)^2 - 12M^2}\right\}^{1/2}, \tag{6}$$

$$n_{ce} = \frac{n_{ce}^0}{\sqrt{6T_{ce}/\mu_e}}\left\{[M^2+(3T_{ce}/\mu_e)+(2\Phi/\mu_e)] \pm \sqrt{[M^2+(3T_{ce}/\mu_e)+(2\Phi/\mu_e)]^2-(12M^2T_{ce}/\mu_e)}\right\}^{1/2}, \tag{7}$$

and

$$n_{he} = \frac{n_{he}^0}{\sqrt{6T_{he}/\mu_e}}\left\{[M^2+(3T_{he}/\mu_e)+(2\Phi/\mu_e)] \pm \sqrt{[M^2+(3T_{he}/\mu_e)+(2\Phi/\mu_e)]^2-(12M^2T_{he}/\mu_e)}\right\}^{1/2}. \tag{8}$$

Rewriting the expressions for the densities (6)–(8) in the form $n_j = n_{j0}(\sqrt{a}+\sqrt{b})$ as in Ghosh *et al.*,[20] the new forms of the expressions for the number densities, respectively, for the ions, and cool and hot electrons, are given by

$$n_i = \frac{1}{2\sqrt{3}}\left\{[(M+\sqrt{3})^2-2\Phi]^{1/2} \pm [(M-\sqrt{3})^2-2\Phi]^{1/2}\right\} \tag{9}$$

$$n_{ce} = \frac{n_{ce}^0}{2\sqrt{3T_{ce}/\mu_e}}\left\{[(M+\sqrt{3T_{ce}/\mu_e})^2+(2\Phi/\mu_e)]^{1/2} \pm [(M-\sqrt{3T_{ce}/\mu_e})^2+(2\Phi/\mu_e)]^{1/2}\right\} \tag{10}$$

and

$$n_{he} = \frac{n_{he}^0}{2\sqrt{3T_{he}/\mu_e}}\left\{[(M+\sqrt{3T_{he}/\mu_e})^2+(2\Phi/\mu_e)]^{1/2} \pm [(M-\sqrt{3T_{he}/\mu_e})^2+(2\Phi/\mu_e)]^{1/2}\right\}. \tag{11}$$

The advantage of expressing the densities in the forms indicated in Eqs. (9)–(11) is that it is very much easier to integrate these expressions to obtain the expression for the Sagdeev potential given later as Eq. (13). Secondly, the consistent choice of the lower "minus" sign in each of Eqs. (9)–(11) yields the correct solution for the densities in that the boundary values stipulated in Eq. (5) are recovered in the limits $\xi \to \pm\infty$. Thirdly, using the form of the Sagdeev potential given later as Eq. (13) which was obtained using the density expressions (9)–(11) makes calculation of the second and third derivatives of $V(\Phi)$ (given later as Eqs. (16) and (17)) very much simpler than using the expression (6) for the Sagdeev potential in Ref. 17 to calculate these.

Choosing the lower sign (minus or "−") in each of the expressions for the densities given by Eqs. (9)–(11) and substituting these in Poisson's equation, ultimately, yields the energy integral like form,

$$\frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 + V(\Phi,M) = 0, \tag{12}$$

whereupon our expression for the Sagdeev potential reads

$$\begin{aligned}
V(\Phi,M) = {} & \frac{1}{6\sqrt{3}}\left\{(M+\sqrt{3})^3 - \left(\sqrt{(M+\sqrt{3})^2-2\Phi}\right)^3\right\} - \frac{1}{6\sqrt{3}}\left\{(M-\sqrt{3})^3 - \left(\sqrt{(M-\sqrt{3})^2-2\Phi}\right)^3\right\} \\
& + \frac{n_{ce}^0}{6\sqrt{3T_{ce}/\mu_e}}\mu_e\left\{(M+\sqrt{3T_{ce}/\mu_e})^3 - \left(\sqrt{(M+\sqrt{3T_{ce}/\mu_e})^2+\frac{2\Phi}{\mu_e}}\right)^3\right\} \\
& - \frac{n_{ce}^0}{6\sqrt{3T_{ce}/\mu_e}}\mu_e\left\{(M-\sqrt{3T_{ce}/\mu_e})^3 - \left(\sqrt{(M-\sqrt{3T_{ce}/\mu_e})^2+\frac{2\Phi}{\mu_e}}\right)^3\right\} \\
& + \frac{n_{he}^0}{6\sqrt{3T_{he}/\mu_e}}\mu_e\left\{(M+\sqrt{3T_{he}/\mu_e})^3 - \left(\sqrt{(M+\sqrt{3T_{he}/\mu_e})^2+\frac{2\Phi}{\mu_e}}\right)^3\right\} \\
& - \frac{n_{he}^0}{6\sqrt{3T_{he}/\mu_e}}\mu_e\left\{(M-\sqrt{3T_{he}/\mu_e})^3 - \left(\sqrt{(M-\sqrt{3T_{he}/\mu_e})^2+\frac{2\Phi}{\mu_e}}\right)^3\right\}.
\end{aligned} \tag{13}$$

This expression for the Sagdeev potential (13) differs from the expression (6) in Ref. 17 because in obtaining Eq. (13), we have used the expressions (9)–(11) with the correct choice of sign for the number densities of the different species rather than using Eqs. (6)–(8) as was the case in Ref. 17.

The requirements for soliton solutions to be supported by Eq. (12) are (i) $V(\Phi) = \mathrm{d}V(\Phi)/\mathrm{d}\Phi = 0$ at $\Phi = 0$, (ii) $(\mathrm{d}^2V(\Phi)/\mathrm{d}\Phi^2)_{\Phi=0} < 0$ (the origin is an unstable fixed point), (iii) $V(\Phi) = 0$ at $\Phi = \Phi_{\text{negative(positive)}}$ where $\Phi = \Phi_{\text{negative(positive)}}$ is a negative (positive) root of $V(\Phi) = 0$ such that $V(\Phi) < 0$ for $\Phi_{\text{negative}} < \Phi < 0$ for negative potential solitons and $0 < \Phi < \Phi_{\text{positive}}$, for positive potential solitons, (iv) $(\mathrm{d}^3V(\Phi)/\mathrm{d}\Phi^3)_{\Phi=0} < 0$ for negative potential solitons and $(\mathrm{d}^3V(\Phi)/\mathrm{d}\Phi^3)_{\Phi=0} > 0$ for positive potential solitons. In addition, a negative (positive) potential soliton solution requires that (v) $(\mathrm{d}V(\Phi)/\mathrm{d}\Phi)_{\Phi=\Phi_{\text{negative}}} < 0$ for negative potential solitons and $(\mathrm{d}V(\Phi)/\mathrm{d}\Phi)_{\Phi=\Phi_{\text{positive}}} > 0$ for positive potential solitons which ensures that a pseudo particle experiences a force in the direction of increasing negative (decreasing positive) values of Φ for negative (positive) potential solitons, so that it is reflected and returns to the origin ($\Phi = 0$). The requirements for a double layer solution is that in addition to the conditions (i)–(iii) for solitons, the requirement (vi) $(\mathrm{d}^2V(\Phi)/\mathrm{d}\Phi^2) < 0$ must be satisfied at $\Phi = \Phi_{\text{negative}}$ (or $\Phi = \Phi_{\text{positive}}$) for a negative (positive) potential double layer.

Although we mention here the small amplitude soliton results, since these are quite useful in describing solutions for solitons of arbitrary amplitude but which are not too large, we must bear in mind that only single polarity soliton structures are predicted by the small amplitude Korteweg-de Vries (KdV) approach and is not reliable when the coexistence of opposite polarity solitons occurs. In the limit of small amplitude, one may Taylor expand $V(\Phi)$ to third order to obtain

$$V(\Phi) \approx C_2\Phi^2 + C_3\Phi^3 \tag{14}$$

for which the small amplitude solution can be written as

$$\Phi = -\left(\frac{C_2}{C_3}\right)\operatorname{sech}^2\left(\sqrt{-\frac{C_2}{2}\xi^2}\right), \tag{15}$$

where $C_2 = \frac{1}{2}\left(\frac{\mathrm{d}^2V(\Phi)}{\mathrm{d}\Phi^2}\right)_{\Phi=0}$ and $C_3 = \frac{1}{6}\left(\frac{\mathrm{d}^3V(\Phi)}{\mathrm{d}\Phi^3}\right)_{\Phi=0}$, where the second and third derivatives of the unapproximated form of the Sagdeev potential (13) are, respectively, given by

$$\left(\frac{\mathrm{d}^2V(\Phi)}{\mathrm{d}\Phi^2}\right)_{\Phi=0} = \frac{1}{[M^2-3]} + \frac{n^0_{ce}}{\mu_e[M^2-(3T_{ce}/\mu_e)]} + \frac{n^0_{he}}{\mu_e[M^2-(3T_{he}/\mu_e)]} \tag{16}$$

and

$$\left(\frac{\mathrm{d}^3V(\Phi)}{\mathrm{d}\Phi^3}\right)_{\Phi=0} = \frac{3[M^2+1]}{[M^2-3]^3} - \frac{3n^0_{ce}[M^2+(T_{ce}/\mu_e)]}{\mu_e^2[M^2-(3T_{ce}/\mu_e)]^3} - \frac{3n^0_{he}[M^2+(T_{he}/\mu_e)]}{\mu_e^2[M^2-(3T_{he}/\mu_e)]^3}. \tag{17}$$

The expression (16) is identical with that given in Ref. 17. The root(s) of Eq. (16) correspond to the critical values of the Mach number, M_{crit}, which is (are) the minimum allowed values of the Mach number which have to be exceeded in order for soliton solutions to be supported by the model. In order to determine the critical values of M, we solve Eq. (16) numerically. For the parameter regions investigated in this paper, two positive roots have been obtained of which the higher value of M_{crit} has been identified as an electron-acoustic mode and the smaller value of M_{crit} as an ion-acoustic mode.[17] Thus, the solution of Eq. (16) corresponding to the smaller (higher) value of M_{crit} provides a lower bound on the permitted Mach number range for which large amplitude ion-acoustic (electron-acoustic) solitons are supported. The existence of a minimum value of the Mach number guarantees that the speed of nonlinear wave structures must exceed the acoustic speed of the corresponding linear wave mode.

We have also included the expression for the third derivative of $V(\Phi)$ evaluated at $\Phi = 0$ given as Eq. (17). The sign of $C_3(M)$ evaluated at $M = M_{\text{crit}}$ dictates what the polarity of the solitons should be in the limit of small amplitude. It is clear from the small amplitude solution (15) that since we must have $C_2 < 0$ for solitons, the sign of $C_3(M_{\text{crit}})$ determines the polarity of solitons having small amplitudes, stated as condition (iv) for solitons given earlier.

Plots of the Sagdeev potential given by Eq. (13) for values of the Mach number which exceed the lower Mach number limit indicate that the amplitudes of large amplitude solitons having positive (negative) potentials increase with increasing values of M. However, this will not continue indefinitely, since a value of M will be reached such that the maximum (or minimum) allowable value of the potential for a positive (negative) potential soliton is attained, such that a soliton will no longer occur for a larger value of M. The question which arises is what limits the occurrence of solitons from the side of high Mach numbers. The focus of the discussion which follows is on how upper Mach number limits on the permitted soliton ranges arise.

For positive values of the potential ($\Phi > 0$), the realization that there is a limit on Φ is clearly apparent from the expression for the number density of the ions given by Eq. (9) but choosing the lower "minus" sign, which reveals that there exists a maximum value of the potential (positive), *viz*, $\Phi_{\max} = (M-\sqrt{3})^2/2$ such that the number density of the ions (9) ceases to be real when $\Phi > \Phi_{\max}$. The limitation on the permitted positive potentials ($\Phi > 0$) of positive potential solitons is, therefore, imposed by the requirement that the ion number density (9) must be real valued. For increasing values of M, positive potential solitons become stronger (increasing amplitudes), but this will not continue indefinitely since the upper limiting value of M which coincides with the limiting value $\Phi = \Phi_{\max}$ will eventually be reached. For values of M for which Φ exceeds $\Phi_{\max}$, Eq. (9) becomes complex valued and a positive root of $V(\Phi)$ which is crucial for a positive potential soliton solution will cease to exist ruling out the possibility of positive potential solitons. Hence, it is clear that the upper Mach number limit imposed by the ion number density having to be real valued applies to large amplitude positive potential ion-acoustic solitons found later.

The situation is more complicated when we consider amplitude restrictions associated with negative potential soliton structures because these will be limited by density constraints imposed by negatively charged plasma constituents and there are two electron species, for which, inertial effects have been included for both. For a small enough value of the negative potential, either the hot or cool electrons could be responsible for the upper limit, since the restriction on the amplitude of the negative potential structures could be imposed by either the cool or hot electron number density becoming complex valued. The number density of the cool electrons given by Eq. (10) with the lower "minus" sign will cease to be real valued if $\Phi < \Phi_{\rm min/cool}$ where the limiting value of the potential imposed by the cool electron species is given by $\Phi_{\rm min/cool} = -\mu_e(M - \sqrt{3T_{ce}/\mu_e})^2/2$. Similarly, the hot electron number density with the choice of the lower "minus" sign in Eq. (11) will not be real valued if $\Phi < \Phi_{\rm min/hot}$ where the limiting value of the potential imposed by the hot electron species is given by $\Phi_{\rm min/hot} = -\mu_e(M - \sqrt{3T_{he}/\mu_e})^2/2$. For values of the Mach number exceeding $M_{\rm crit}$, negative potential solitons will become stronger (increasing amplitudes) with increasing values of M that exceed the critical value $M_{\rm crit}$ which is the lower limit. This will not continue indefinitely, since the upper limit on M will eventually be reached, which could coincide with either $\Phi_{\rm min/cool}$ (or $\Phi_{\rm min/hot}$). For a value of M which exceeds the upper Mach number limit such that $\Phi < \Phi_{\rm min/cool}$ (or $\Phi < \Phi_{\rm min/hot}$), the number density of the cool electrons (or hot electrons) becomes complex valued such that a negative root of $V(\Phi)$ will not occur and a negative potential soliton solution is no longer possible. In order to ascertain which of the aforementioned scenarios is applicable, whether the cool or hot electrons will be responsible for limiting the existence of negative potential soliton structures, one has to turn to numerical considerations of $V(\Phi)$ given by Eq. (13). The restrictions pertaining to the number densities of the cool and hot electrons having to remain real valued can limit the occurrence of large amplitude negative potential solitons and it will be seen later that these restrictions will apply to large amplitude electron-acoustic solitons for which existence domains are discussed in detail in II.

The existence of upper Mach number limits for solitons are not only restricted to density considerations of the charged particle constituents. It is also well known that double layers for which $dV(\Phi)/d\Phi = 0$ coincides with a positive (negative) root of $V(\Phi)$ for positive potential (negative potential) double layers can also limit the existence domains of solitons.[9–11] It is not obvious from the form of $V(\Phi)$ whether a double layer will or will not occur. We have to rely on numerical considerations of $V(\Phi)$ in order to establish whether double layer solutions are possible. In our investigations, although double layers were not found to limit the occurrence of ion-acoustic solitons, both negative and positive potential double layers have been found to limit the existence domains of negative and positive potential electron-acoustic solitons in certain regions of parameter space as discussed in II.

III. MODEL WITH BOLTZMANN HOT ELECTRONS

In this section, we consider the model of Mace *et al.*[18] where the inertia (and pressure) of the ions and cool electrons is retained but inertia of the hot electrons is not taken into consideration. The Eqs. (1)–(3) are all still valid for the ions and the cool electrons except that the number density of the hot electrons, which are assumed to be Boltzmann distributed, is now given by the normalized expression

$$n_{he} = n_{he}^0 \exp\left(\frac{\Phi}{T_{he}}\right). \quad (18)$$

Substituting the same expressions (9) for the number density of the ions and Eq. (10) for the number density of the cool electrons (having chosen the lower "minus" sign in both of Eqs. (9) and (10)) but now using the expression (18) for Boltzmann hot electrons in Eq. (4) yields for the Sagdeev potential the expression given by

$$\begin{aligned} V(\Phi, M) = {} & \frac{1}{6\sqrt{3}}\left\{(M+\sqrt{3})^3 - \left(\sqrt{(M+\sqrt{3})^2 - 2\Phi}\right)^3\right\} - \frac{1}{6\sqrt{3}}\left\{(M-\sqrt{3})^3 - \left(\sqrt{(M-\sqrt{3})^2 - 2\Phi}\right)^3\right\} \\ & + \frac{n_{ce}^0}{6\sqrt{3T_{ce}/\mu_e}}\mu_e\left\{(M+\sqrt{3T_{ce}/\mu_e})^3 - \left(\sqrt{(M+\sqrt{3T_{ce}/\mu_e})^2 + \frac{2\Phi}{\mu_e}}\right)^3\right\} \\ & - \frac{n_{ce}^0}{6\sqrt{3T_{ce}/\mu_e}}\mu_e\left\{(M-\sqrt{3T_{ce}/\mu_e})^3 - \left(\sqrt{(M-\sqrt{3T_{ce}/\mu_e})^2 + \frac{2\Phi}{\mu_e}}\right)^3\right\} + n_{he}^0 T_{he}\left[1 - \exp\left(\frac{\Phi}{T_{he}}\right)\right]. \end{aligned} \quad (19)$$

The second and third derivatives of $V(\Phi)$ given by Eq. (19) evaluated at $\Phi = 0$ is, now, respectively given by

$$\left(\frac{d^2V(\Phi)}{d\Phi^2}\right)_{\Phi=0} = \frac{1}{[M^2-3]} + \frac{n_{ce}^0}{\mu_e[M^2-(3T_{ce}/\mu_e)]} - \frac{n_{he}^0}{T_{he}} \quad (20)$$

and

$$\left(\frac{d^3V(\Phi)}{d\Phi^3}\right)_{\Phi=0} = \frac{3[M^2+1]}{[M^2-3]^3} - \frac{3n_{ce}^0[M^2+(T_{ce}/\mu_e)]}{\mu_e^2[M^2-(3T_{ce}/\mu_e)]^3} - \frac{n_{he}^0}{T_{he}^2}. \quad (21)$$

The restrictions on the attainable amplitudes of positive and negative potential soliton structures, respectively, imposed by the constraint that the number density of the ions and the cool electrons must remain real valued as discussed in Sec. II still apply, however, a layer of complexity is removed when the hot electrons are Boltzmann distributed, since now there are no restrictions on the amplitudes of negative potential solitons imposed by the hot electrons, since the number density of the hot electrons (18) can never become complex valued due to the exponential dependence of the number density of the hot electrons on the potential.

IV. NUMERICAL RESULTS AND DISCUSSION

We initially investigate the existence of large amplitude ion-acoustic solitons, which are supported by the model of Sec. II for which inertia and pressure is included for all species.[17] The Mach number ranges supporting the existence of large amplitude ion-acoustic solitons are indicated as a function of n_{ce0}/n_{i0} in Figure 1, starting at $n_{ce0}/n_{i0} = 0.05$ and terminating at $n_{ce0}/n_{i0} = 1.0$, where, n_{ce0}/n_{i0} is the cool electron number density expressed as a fraction of the ion or total electron number density given by $n_{i0} = n_{ce0} + n_{he0}$. We recall that n_{ce0}, n_{he0}, and n_{i0}, respectively, denote the equilibrium number densities of the cool electrons, hot electrons, and ions. The lower curve (—) for the critical Mach number, $M_{\rm crit}(n_{ce0}/n_{i0})$, is obtained by solving Eq. (16) and choosing the smaller of the two positive roots. Following the ideas in Ref. 16, the upper Mach number limiting curve denoted by $(\cdots)$ in Figure 1 is generated by solving $V(\Phi_{\max}) = 0$ for M as a limiting case for the requirement that $V(\Phi) > 0$ for $\Phi > \Phi_{\max}$. We recall from the discussion in Sec. II that $\Phi_{\max} = (M - \sqrt{3})^2/2$ is the maximum permitted value of the potential (positive) such that the ion number density given by Eq. (9) will become complex valued for $\Phi > \Phi_{\max}$. It becomes clear from the figure that for any fixed value of n_{ce0}/n_{i0}, the choice of value for the Mach number for which a large amplitude ion-acoustic soliton occurs is restricted and must lie within the allowed ranges depicted in Figure 1.

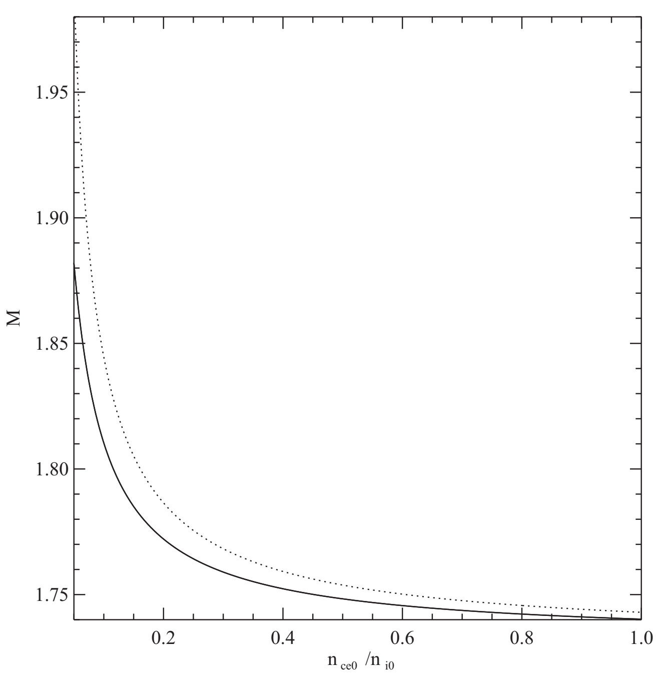

FIG. 1. Existence domains of ion-acoustic solitons shown as a function of the normalized cool electron number density where the curve (–) denotes $M_{\rm crit}(n_{ce0}/n_{i0})$ and $(\cdots)$ denotes the maximum allowed value of the Mach number corresponding to the ion number density given by the expression (9) (with the choice of the lower "minus" sign) being real valued. The fixed parameters are $\mu_e = 1/1836$, $T_{ce}/T_i = 0.01$, and $T_{he}/T_i = 5$.

The existence of lower and upper limiting values of the Mach number restrict the choice of the permitted value of M for which large amplitude ion-acoustic solitons are possible. This becomes clearly apparent from plots of the Sagdeev potential (13) in Figure 2 for $n_{ce0}/n_{i0} = 0.3$ and the other fixed parameters indicated for Figure 1. The curve (–) denotes a plot of $V(\Phi)$ given by Eq. (13) corresponding to the critical value of the Mach number, *viz*, $M_{\rm crit} = 1.75897$ for $n_{ce0}/n_{i0} = 0.3$. This lower limiting value of M coincides with the point corresponding to $n_{ce0}/n_{i0} = 0.3$ on the lower limiting Mach number curve denoted by (–) in Figure 1. A valid ion-acoustic soliton solution requires that this critical value of the Mach number ($M = 1.75897$) must be exceeded, so that $V(\Phi)$ not only has a local maximum at the origin but there must also exist a positive root of $V(\Phi)$, *viz*, $\Phi_{\rm positive}$ such that $V(\Phi = \Phi_{\rm positive}) = 0$ and $(dV(\Phi)/d\Phi)_{\Phi=\Phi_{\rm positive}} > 0$ at the position of the root. The latter requirement guarantees that when the pseudo-particle leaves the origin ($\Phi = 0$), there is a force acting on it so that it can return to the origin ($\Phi = 0$) as is necessary for a soliton solution but not for a double layer. The behaviour of $V(\Phi)$ as demonstrated by the curve denoted by $(\cdots)$ for $M = 1.765$ in Figure 2 satisfies all the requirements for a soliton solution. For a higher value of the Mach number, *viz*, $M = 1.767$, we observe that the amplitude of the nonlinear ion-acoustic soliton structure increases as can be inferred from the higher positive root of $V(\Phi)$ for $M = 1.767$ denoted by (- -) in Figure 2 in comparison with the positive root of $V(\Phi)$ for $M = 1.765$ $(\cdots)$. We observe in

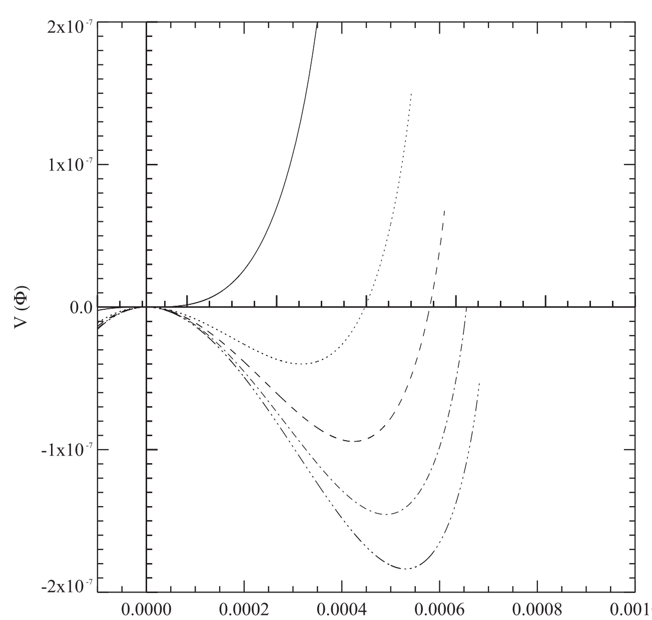

FIG. 2. Sagdeev potential profiles for $M = 1.75897$ (–), $M = 1.765$ $(\cdots)$, $M = 1.767$ (– –), 1.76825 (– · –), and 1.769 (– · · · –). The fixed parameters are $\mu_e = 1/1836$, $T_{ce}/T_i = 0.01$, $T_{he}/T_i = 5$, and $n_{ce0}/n_{i0} = 0.3$.

Figure 2 that for increasing values of the Mach number which exceed the value M_{crit}, ion-acoustic solitons are seen to become stronger (increasingly larger values of Φ_{positive}). This will not continue indefinitely, since the upper Mach number limit will eventually be reached for which the upper limiting curve of the Sagdeev potential denoted by $(- \cdot -)$ in Figure 2 corresponding to the value $M = 1.76825$ is obtained. This upper limiting value of M, *viz*, $M = 1.76825$ coincides precisely with the limiting value of Φ, *viz*, $\Phi_{\max} = (M - \sqrt{3})^2/2$, for which Eq. (9) is real valued. It is important to note that the behaviour of the limiting curve of $V(\Phi)$ for $M = 1.76825$ does not satisfy the requirement (iii) for a positive potential soliton solution stipulated in Sec. II, *viz*, $(\mathrm{d}V(\Phi)/\mathrm{d}\Phi)_{\Phi=\Phi_{\text{positive}}} > 0$. This upper limiting value of the Mach number ($M = 1.76825$), which lies on the upper Mach number limiting curve denoted by $(\cdots)$ in Figure 1 corresponding to the fixed value $n_{ce0}/n_{i0} = 0.3$, does not yield an ion-acoustic soliton solution. A value for M which exceeds the upper Mach number limit such that $\Phi > \Phi_{\max}$ does not yield an ion-acoustic soliton solution as clearly demonstrated by the behaviour of the curve denoted by $(- \cdots -)$ corresponding to the value $M = 1.769$ in Figure 2, since $V(\Phi)$ no longer has a positive root. For the parameters considered here, only positive potential ion-acoustic solitons are found to be possible. This is found to be consistent with the predictions of the theory of small amplitude solitons, since for the parameters of Figures 1 and 2, the sign of $C_3(M_{\text{crit}})$, Eq. (17) is found to be positive predicting positive polarities for solitons in the limit of small amplitude as is evident from Eq. (15).

One can appreciate the usefulness of a figure such as Figure 1 which depicts existence domains of large amplitude ion-acoustic solitons, since the Mach numbers ranges which support the existence of ion-acoustic solitons can quickly be ascertained for any choice of the value of n_{ce0}/n_{i0} chosen from the wide range of values depicted in Figure 1. In generating Figure 1, we have relied on physical insight as to why upper Mach number limits exist for ion-acoustic solitons rather than having to resort to producing a very large number of plots of Sagdeev potential profiles similar to Figure 2 for each value of n_{ce0}/n_{i0} in order to be able to identify the upper limits of the Mach number ranges supporting the occurrence of large amplitude ion-acoustic solitons.

Fixing the concentration of the cool electrons, *viz*, $n_{ce0}/n_{i0} = 0.3$, but varying T_{he}/T_i, the Mach number ranges for which the existence of large amplitude ion-acoustic solitons is supported are shown in Figure 3. For each value of T_{he}/T_i, the lower Mach number and upper Mach number limits, respectively, coincide with points on the lower and upper limiting curves denoted by (–) and $(\cdots)$ in Figure 3. We observe a rapid increase in the lower and upper Mach number limits with increasing values of T_{he}/T_i starting from $T_{he}/T_i = 0.1$ to $T_{he}/T_i = 2$, but the lower and upper Mach number limits appear to be insensitive to changes in T_{he}/T_i beyond the value of $T_{he}/T_i = 2$. Although we have depicted existence domains of ion-acoustic solitons starting from very small values of the temperature ratio $T_{he}/T_i = 0.1 < 1$, we must bear in mind that linear ion-acoustic waves are strongly damped unless $T_e/T_i \gg 1$ in plasmas composed of ions and a single electron species, so one must exercise caution when discussing the existence of nonlinear ion-acoustic structures if the temperature ratio T_{he}/T_i is not very much greater than unity, especially when there are no beams present, since Landau damping rates of linear ion-acoustic waves are large, ruling out the possibility that ion-acoustic solitons will occur.

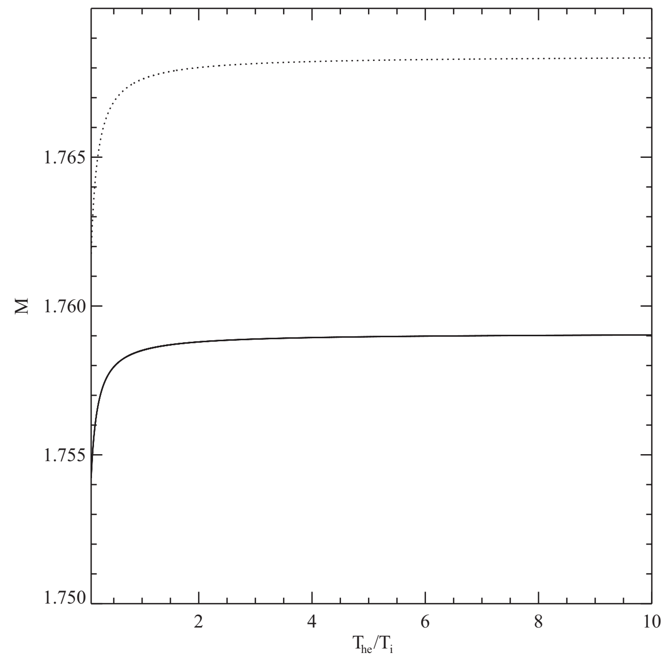

FIG. 3. Existence domains of ion-acoustic solitons shown as a function of the normalized hot electron temperature T_{he}/T_i where the curve (–) denotes $M_{\text{crit}}(T_{he}/T_i)$ and $(\cdots)$ denotes the maximum allowed value of the Mach number corresponding to the ion number density given by the expression (9) (with the choice of the lower "minus" sign) being real valued. The fixed parameters are $\mu_e = 1/1836$, $n_{ce0}/n_{i0} = 0.3$, and $T_{ce}/T_i = 0.01$.

Our investigations up to this point reveal that ion-acoustic solitons having only positive potentials can be supported. Our curiosity led us to wonder whether negative potential ion-acoustic soliton structures are at all possible for the model of Sec. II for which inertia has been included for all species. Our findings reveal that negative potential ion-acoustic solitons are possible when the pressure of the cool electrons is so negligible to the extent where this species can be regarded as cold ($T_{ec} = 0$). These are depicted in Figure 4. The negative polarities of the solitons which occur for $1.73077 < M \le 1.732050807$ in Figure 4 are consistent with the negative sign of $C_3(M_{\text{crit}})$ for small amplitude solitons, which is obtained using the expressions (16) and (17). All the conditions (i) to (v) stipulated in Sec. II are satisfied for M values in the range $1.73077 < M \le 1.732050807$, which confirms that the nonlinear structures shown in Figure 4 are indeed solitons. Going to higher values of the Mach number which exceed $M = 1.732050807$, our findings reveal that some of the conditions (i) to (v) stipulated in Sec. II are violated including the condition (ii) for a soliton, since the second derivative of $V(\Phi)$ is positive valued, proving that $V(\Phi, M)$ has a local minimum rather than a local maximum at $\Phi = 0$. Based on these observations, our results confirm that the existence of solitons terminates at the value $M = 1.732050807$ (last curve denoted by (– –) in Figure 4) and are not possible for values of M, which exceed the value

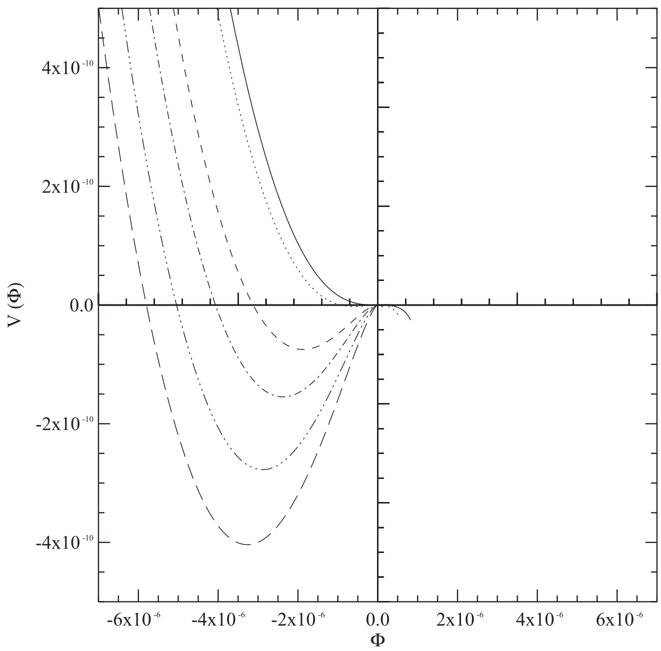

FIG. 4. Sagdeev potential profiles for $M = 1.73077$ (—), $M = 1.731$ (· · ·), $M = 1.7315$ (- -), 1.7317 (– · –), 1.7319 (– · · · –), and $M = 1.732050807$ (–-). The fixed parameters are $\mu_e = 1/1836$, $n_{ce0}/n_{i0} = 0.3$, $T_{ce}/T_i = 0.0001$, and $T_{he}/T_i = 5$.

$M = 1.732050807$. The change in shape of $V(\Phi)$ which favours the existence of solitons ($1.73077 < M \leq 1.732050807$ in Figure 4) to a shape that is atypical for solitons (not shown in Figure 4) is observed to be quite sudden and occurs when M exceeds 1.732050807 for the parameters mentioned for Figure 4. A very rough calculation indicates that the minimum permitted value of the negative potential corresponding to the number density of the cool electrons (10) (with the choice of the lower "–" sign) being real valued is $\approx -2.66 \times 10^{-4}$. This limit is not relevant for the negative potential ion-acoustic solitons shown in Figure 4, since solitons cease to exist long before this limit can be reached.

We do not include our results for the model of Sec. III for which the hot electrons are Boltzmann distributed. When inertial effects of the hot electrons is not taken into consideration, the lower and upper Mach number limits for ion-acoustic solitons are only very slightly reduced to the extent that the figure depicting existence domains of positive potential ion-acoustic solitons using the model of Sec. III having Boltzmann hot electrons would appear identical to Figure 1, which, we recall, was generated using the model of Sec. II which includes inertial effects of the hot electrons. Considering negative potential ion-acoustic solitons, the results for the model of Sec. III having Boltzmann distributed hot electrons appears identical to the Sagdeev potential profiles depicted in Figure 4, since the effect of neglecting inertia of the hot electrons is only to reduce the values of the lower and upper Mach number limits but only very slightly. There are no significant differences in our results for large amplitude ion-acoustic solitons if either the model of Sec. II or the model of Sec. III is chosen. The differences in our findings arising from including hot electron inertia as opposed to neglecting hot electron inertia are found to be substantially more significant for large amplitude electron-acoustic solitons and qualitative differences are observed as discussed in our companion paper II.

V. CONCLUSIONS

The existence of large amplitude ion-acoustic solitons has been investigated for a three-component plasma composed of ions, cool electrons, and hot electrons, not only for the model of Lakhina *et al.*,[17] for which, inertia and pressure has been retained for all species, but the effect of a neglect of inertial effects of the hot electrons in accord with the model of Mace *et al.*[18] has also been investigated. We recall that in Ref. 17, the focus was mainly on identifying the lower limits in Mach number space, which must be exceeded for low-frequency ion-acoustic and high-frequency electron-acoustic solitons to be supported. Focusing here only on the low-frequency ion-acoustic soliton structures, we have significantly extended the scope of the findings in Ref. 17 by not only determining the upper limits of the Mach number ranges supporting the existence of large amplitude ion-acoustic solitons in different regions of parameter space where they occur, but we also provide reasons as to why these upper Mach number limits exist for ion-acoustic solitons. Taking both the lower and upper Mach number limits for large amplitude ion-acoustic solitons into consideration, we have presented the permitted Mach number ranges supporting the existence of large amplitude ion-acoustic solitons for much broader regions in parameter space than those investigated in Ref. 17. Consistent with the findings of Lakhina *et al.*,[17] only positive potential ion-acoustic solitons are found to be supported in regions of parameter space where the cool electrons are not cold but assumed to have finite but small pressure. Our findings indicate that the upper Mach number limits for large amplitude ion-acoustic solitons having positive potential coincide with the maximum permitted values of the potential (positive) imposed by the number density of the ions having to be real valued. For values of the Mach number which exceed the upper limit, a positive root of the Sagdeev potential is no longer possible ruling out the possibility for positive potential solitons.

Surprising for us, having widened the scope of the study in Ref. 17 to also include parameter regions where the pressure of the cool electrons is so negligible that this species can be regarded as cold ($T_{ce} = 0.0$), our findings reveal that negative potential ion-acoustic solitons are possible. These are found to have much smaller amplitudes than the positive potential ion-acoustic solutions found earlier and the upper Mach number limit for these negative potential ion-acoustic soliton structures is not imposed by the constraint relating to the number density of the cool electrons having to remain real valued, but arises because the Sagdeev potential $V(\Phi)$ no longer has the shape which is in accord with the requirements for a soliton when the Mach number exceeds the upper limit.

It is interesting to point out that the existence regions of the ion-acoustic solitons having positive or negative potentials found for the parameters considered here is not significantly altered when the inertia of the hot electrons is neglected as in the model of Mace *et al*.[18] This may be due to the fact that on the time scale of ion-acoustic solitons, there is enough time for the temperature of the hot electrons to become equalized to T_h to establish the Boltzmann equilibrium.

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[1]R. E. Ergun., C. W. Carlson, J. P. McFadden, F. S. Mozer, G. T. Delory, W. Peria, C. C. Chaston, M. Temerin, I. Roth, L. Muschietti, R. Elphic, R. Strangeway, R. Pfaff, C. A. Cattell, D. Klumpar, E. Shelley, W. Peterson, E. Moebius, and L. Kistler, Geophys. Res. Lett. **25**, 2041, doi:10.1029/98GL00636 (1998).
[2]J. R. Franz, P. M. Kintner, and J. S. Pickett, Geophys. Res. Lett. **25**, 1277, doi:10.1029/98GL50870 (1998).
[3]S. D. Bale, P. J. Kellogg, D. E. Larson, R. P. Lin, K. Goetz, and R. P. Lepping, Geophys. Res. Lett. **25**, 2929, doi:10.1029/98GL02111 (1998).
[4]H. Matsumoto, H. Kojima, T. Miyatake, Y. Omura, M. Okada, I. Nagano, and M. Tsutsui, Geophys. Res. Lett. **21**, 2915, doi:10.1029/94GL01284 (1994).
[5]J. S. Pickett, J. D. Menietti, D. A. Gurnett, B. Tsurutani, P. M. Kintner, E. Klatt, and A. Balogh, Nonlinear Process. Geophys. **10**, 3 (2003).
[6]M. Temerin, K. Cerny, W. Lotko, and F. S. Mozer, Phys. Rev. Lett. **48**, 1175 (1982).
[7]H. E. J. Koskinen, R. Lundin, and B. Holback, J. Geophys. Res. **95**, 5921, doi:10.1029/JA095iA05p05921 (1990).
[8]A. Mälkki, A. I. Eriksson, P. O. Dovner, R. Boström, B. Holback, G. Holmgren, and H. E. J. Koskinen, J. Geophys. Res. **98**, 15521, doi:10.1029/93JA01312 (1993).
[9]S. Baboolal, R. Bharuthram, and M. A. Hellberg, J. Plasma Phys. **40**, 163 (1988).
[10]S. Baboolal, R. Bharuthram, and M. A. Hellberg, J. Plasma Phys. **44**, 1 (1990).
[11]R. Bharuthram and P. K. Shukla, Phys. Fluids **29**, 3214 (1986).
[12]M. A. Hellberg and F. Verheest, Phys. Plasmas **15**, 062307 (2008).
[13]J. F. McKenzie, Phys. Plasmas **9**, 800 (2002).
[14]J. F. McKenzie, J. Plasma Phys. **69**, 199 (2003).
[15]J. F. McKenzie and T. B. Doyle, New J. Phys. **5**, 26 (2003).
[16]F. Verheest and M. A. Hellberg, J. Plasma Phys. **76**, 277 (2010).
[17]G. S. Lakhina, A. P. Kakad, S. V. Singh, and F. Verheest, Phys. Plasmas **15**, 062903 (2008).
[18]R. L. Mace, S. Baboolal, R. Bharuthram, and M. A. Hellberg, J. Plasma Phys. **45**, 323 (1991).
[19]C. A. Mendoza-Briceño, S. M. Russel, and A. A. Mamun, Planet Space Sci. **48**, 599 (2000).
[20]S. S. Ghosh, K. K. Ghosh, and A. N. Sekar Iyengar, Phys. Plasmas **3**, 3939 (1996).